A Geometric Approach to the Gibbs–Appell Equations in Lagrangian Mechanics

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Abstract

We present a geometrical derivation of the Gibbs–Appell equations in constrained Lagrangian mechanics using the affine structures existing in the spaces of velocities and accelerations. The theoretical basis is the Gauss principle of least constraint, and the necessary conditions for the minimum expressed in terms of the Gibbs function give rise to the Gibbs–Appell equations. We give a second-order formulation of the constraint equation, and using the concept of Moore-Penrose inverse, we provide an explicit intrinsic construction of the projector on the space of the effective or non-ideal forces. The Gibbs–Appell equations are expressed in an intrinsic way in terms of the fibre derivative of the Gibbs function and the effective part of the applied force.

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1 Introduction

Among the best known formulations of the equations of motion in mechanics, the Lagrange approach is particularly well suited to deal with the case of holonomic systems, where the space of states is the tangent bundle $TM$ of the configuration manifold $M$. Lagrange’s equations expressed in the Lagrangian coordinates $(q, \dot{q})$ are invariant under arbitrary changes of the coordinates $q$ required in order to specify the configurations; moreover, coordinates for velocities other than the derivatives of a system of independent coordinates of position are also allowed, but the equations of motion must be modified and acquire a more complicated form (equations of Hamel or Poincaré) in terms of the so called quasi-velocities [7, 12]. The extension of Lagrange’s equations to non-holonomic systems, where the space of states is no longer $TM$ but one of its submanifolds (see e.g. [13, 14]) use the method of Lagrange multipliers together with d’Alembert’s principle to give rise to the equations of motion for the system [8], the undetermined Lagrange multipliers representing the constraint forces. A variational approach to systems with linear velocity constraints with an augmented Lagrangian, where the constraints are added to the given Lagrangian multiplied by some appropriated Lagrange multipliers, is not appropriate for non-holonomic mechanics but for ‘Vakonomic dynamics’ (variational axiomatic kind dynamics) [16, 25, 32, 35] which is useful in control theory.

An alternative, and more general, method is also possible, in which the constraints, no matter what their character (linear or not), are used to select the independent velocities in a number lower than the number of independent coordinates $q$. The method was discovered independently by Gibbs [21] and Appell [1], and the fundamental equations are known as the Gibbs–Appell equations. As is well known, these equations can be obtained from the Gauss principle of least constraint [34, 41].

In this approach to the equations of motion, the use of quasi-velocities is quite natural, and the form of the equations is unaffected by general changes of coordinates of configurations and velocities. In words of Pars [34], “the Gibbs–Appell equations provide what is probably the simplest and most comprehensive form of the equations of motion so far discovered. They are of superlatively simple form, they apply with equal facility to holonomic and non-holonomic systems alike, and quasi-coordinates may be used freely.” In these equations, as in Lagrange’s equations in the holonomic case, the constraint forces do not appear, only (the effective part of) the applied forces do appear.

In spite of these advantages, the equations are not usually treated in textbooks in theoretical and applied mechanics, although there are exceptions, as for instance [18, 20, 24, 34, 40, 41]. Moreover, there are several recent works on the geometry
of the main field of application of the Gibbs–Appell equations, namely the geometry of non-holonomic systems, from different perspectives [6, 7, 15, 22, 26, 27], but only a few detailed works on the geometric approach to the equations (see [29, 39], and also [3, 4] for an approach worked mainly in local coordinates but giving geometrical pictures of some concepts and results). We believe that this subject deserves more attention and our article is intended to provide a geometrical framework in which the Gibbs–Appell equations for constrained Lagrangian systems are derived, step by step, in an intrinsic manner. Actually the use of modern tools of Differential Geometry allows us to present the concepts in a manner independent of coordinates, which are only locally defined, and moreover, in such a coordinate free presentation the theory is ready for generalization to infinite-dimensional cases. The interplay between geometry and physics has been shown beyond doubt to be very useful during the past few years and has provided not only an adequate framework for a better understanding of known properties, but it has also suggested new problems and given new perspectives, allowing us to develop simpler methods for dealing with many problems, and to extend the framework of application of other models.

The plan of the article is as follows. In the next section we review the basic geometric structures on the spaces of 1-jets $TM$ and 2-jets $T^2M$ and their expressions in local coordinates. We insist on the fact that the main geometric objects we need can be seen as affine morphisms between affine bundles.

The equations of the unconstrained motion of a regular Lagrangian system in the desired geometric form are established in section 3. The underlying affine structures allow us to represent a motion of the mechanical system by a section $\gamma$ of the affine bundle $T^2M \to TM$ instead of the usual representation by a (equivalent) second-order vector field $\Gamma \in \mathfrak{X}(TM)$. Moreover, every motion $\gamma$ is referred to a particular motion $\gamma_0$ taken as a reference motion; the actual motion can be seen thus as a deviation with respect to $\gamma_0$. Then we introduce the concept of (Lagrangian) force $F$, which appears as a natural concept. In these terms, the equation of motion can be written as a Newtonian-type one for the deviation $\dot{\gamma} - \dot{\gamma}_0$, which represents the vector acceleration of the $\gamma$-motion with respect to the $\gamma_0$-motion.

Section 4 is devoted to analysis of constrained systems and their motions in geometric terms. We give an intrinsic and useful characterization of the admissible accelerations as an affine bundle with underlying vector bundle the space of ‘virtual displacements’, which, in our approach, appear as vertical vectors in $TTM$ belonging to the kernel of the linear part $[dTf]$ of the total time derivative of the constraint function $f$; they relate two accelerations over one and the same state. As usual, the equation of motion is obtained by introducing a ‘constraint force’ which produces a deformation of the Euler–Lagrange equation. The true motion $\gamma$ is the solution
of both the ‘second-order’ constraint equation and the deformed Euler–Lagrange equation, although at this level, the solution is not completely determined, the indetermination being a field of virtual displacements which depends on the behaviour of the constraint force under virtual displacements.

In section 5 we specify to the case of ideal constraints and find that the vector acceleration $\dot{\gamma}_R$ of the true motion $\gamma$ with respect to the unconstrained or released motion $\gamma_R$, is the one having the minimal value of the length defined by the Hessian of the Lagrangian function; in other words, the Gauss principle of Least Constraint holds, and vice versa. Now, the key point is that, assuming this principle as the starting point, we are led to the problem of minimising the Gauss Constraint function restricted to the space of admissible accelerations. As is well known, the necessary conditions for the minimum expressed in terms of the Gibbs function give rise to the Gibbs–Appell equations for $\gamma$. This is worked out in section 6.

In our treatment, the Gibbs–Appell equations are expressed in terms of the fiber derivative of the Gibbs function and the effective force, which is the non-ideal part of the ‘full’ force $F$. We provide an explicit intrinsic construction of the projector on the subspace of non-ideal forces in terms of the Hessian of the Lagrangian, and the linear operator $[dTf]$ and its Moore–Penrose inverse.

Finally, in section 7 we include several examples of the use of the Gibbs–Appell equations in local coordinates, especially with quasi-velocities.

2 Some definitions and notation

The basic concepts and techniques concerning the theory of jet bundles may be found in many books, such as [38]. Here we review and complete the geometric constructions given in [10].

The configuration space of an autonomous $m$-dimensional system is the manifold $M$, with tangent bundle projection $\tau_M$. The spaces of ‘velocities’ or ‘(kinematical) states’ and ‘accelerations’ are, respectively, the 1-jet bundle $TM$ and the 2-jet bundle $T^2M$ of curves of $M$. The $k$-jet, $k = 1, 2$, of the (local) curve $\sigma : \mathbb{R} \to M$ at the point $x = \sigma(0)$ is denoted by $j^k_x$\sigma. There is a natural projection $\mu : T^2M \to TM$ given by $\mu \circ j^2_x = j^1_x$, where $j^k_x$\sigma denotes the $k$-jet prolongation of $\sigma$.

The space $T^2M$ can be considered as an immersed submanifold of $TTM$ through the immersion $T : T^2M \to TTM$ that assigns to the 2-jet $a = j^2_x$\sigma the vector tangent to the curve 1-jet prolongation $j^1_x$\sigma of the base curve $\sigma$ at $z = j^1_x$\sigma. By definition, $T$ satisfies the second-order condition $\tau_{TM}(T(a)) = \tau_{TM}(T(a))$ and consequently $\tau_{TM} \circ T = \mu$, that is, $T$ is a vector field along the projection $\mu$. 


A very important fact is that \( \mu : T^2M \to TM \) is an affine bundle whose associated vectorial bundle is the vertical bundle to \( \tau_M \), denoted \( \pi : V\tau_M \to TM \), where \( V\tau_M = \text{Ker}(\tau_M^*) \) and \( \pi = \tau_{TM}|_{V\tau_M} \). Thus, the accelerations over one and the same kinematical state \( z \in TM \) form an affine space \( \mu^{-1}(z) \) modelled on \( Vz\tau_M \). The \( \tau_M \)-vertical vector which corresponds to the couple of points \( a, a' \in \mu^{-1}(z) \) is the vector \( T(a') - T(a) \in Vz\tau_M \), that we denote by \( \overrightarrow{aa'} \).

2.1 Some remarkable structures related to a function \( f \in C^\infty(TM) \).

Every real function \( f \in C^\infty(TM) \) has its ‘total time derivative’, namely the function \( d_Tf = i_Tdf \in C^\infty(T^2M) \), commonly denoted by \( \dot{f} \) or \( df/dt \), which represents the variation of \( f \) along the 1-jet prolongation of the base curves \( \sigma(t) \) (see [10, 11]). By definition \( d_Tf(a) = \langle df(z), T(a) \rangle \), and consequently \( d_Tf(a') - d_Tf(a) = \langle df(z), \overrightarrow{aa'} \rangle \), \( a, a' \in \mu^{-1}(z) \); here \( \langle ., . \rangle \) denotes the pairing of a covector and a vector. Thus, the total time derivative \( d_Tf \) can equivalently be viewed as the fibre bundle morphism \( T^2M \ni a \to (\mu(a), d_Tf(a)) \in TM \times \mathbb{R} \) of the bundle \( \mu \) over the trivial bundle \( pr_1 : TM \times \mathbb{R} \to TM \); it is an affine morphism whose linear part \( [d_Tf] : V\tau_M \to TM \times \mathbb{R} \) is the ‘vertical differential’ of \( f \), that is, \( [d_Tf] \cdot w = (\pi(w), \langle df(\pi(w)), w \rangle) \).

The derivation \( d_T \) is easily extended to the case of a vector-valued function \( f \in C^\infty(TM, \mathbb{R}^r) : d_Tf \in C^\infty(T^2M, \mathbb{R}^r) \) is defined as

\[
d_Tf(a) := f_{sz} \cdot T(a), \quad a \in \mu^{-1}(z);
\]

consequently, as in the case \( r = 1 \), we have the obvious relation \( d_Tf(a') - d_Tf(a) = f_{sz} \cdot \overrightarrow{aa'} \) and \( d_Tf \) is considered to be the affine morphism of \( T^2M \) over \( TM \times \mathbb{R}^r \) whose linear part \([d_Tf]\) is determined by the restriction of \( f \), to vertical vectors.

Other relevant structures are provided by the first and second fibre derivatives of \( f \) [23], \( \mathcal{F}f : TM \to T^*M \) and \( \mathcal{F}^2f : TM \to S^2(TM) \), respectively, given by

\[
\mathcal{F}f(z) := Df_{\tau_M(z)}(z), \quad \mathcal{F}^2f(z) := D^2f_{\tau_M(z)}(z),
\]

where \( z \in TM \), \( f_x \) means \( f|_{T_xM} \), the symbol \( D \) stands for the ordinary derivative, and \( S^2(TM) \) represents the bundle of bilinear symmetric functions of \( TM \). The second fibre derivative induces the bundle morphism \( \mathcal{H}f : TM \to S^2(V\tau_M) \) given by

\[
\mathcal{H}f(z) \cdot (\xi^V(z, v), \xi^V(z, v')) = \mathcal{F}^2f(z) \cdot (v, v'), \quad v, v' \in T_{\tau_M(z)}M,
\]

where \( \xi^V(z, v) \in Vz\tau_M \) is the vertical lift of \( v \in T_{\tau_M(z)}M \) to a vector in \( T_zTM \), see e.g. [9, 17]. With this interpretation, \( \mathcal{H}f \) is called the Hessian map of \( f \). The induced linear morphism between the vertical bundle \( \pi : V\tau_M \to TM \) and its dual \( \pi^* \) is denoted by \( \widehat{\mathcal{H}}f \) and called the Hessian of \( f \) too.
We have also the *Euler–Lagrange form* of \( f, \delta f \in \bigwedge^1(T^2 M) \) defined by

\[
\delta f := i_T d(S^* df) + \mu^* dE_f,
\]

where \( S^* \) is the dual operator of the *vertical endomorphism* \( S \) of \( T(TM) \), \( E_f = \Delta(f) - f \) the *energy* of \( f \) and \( \Delta \in \mathfrak{X}(TM) \) the Liouville field \([9, 17]\).

The Euler–Lagrange form of \( f \) determines a morphism of bundles over \( TM \), denoted by \( \tilde{\delta} f : T^2 M \to V^* \tau_M \) and given by

\[
\langle \tilde{\delta} f(a), \xi^V(z,v) \rangle := \langle \delta f(a), W \rangle, \quad a \in \mu^{-1}(z),
\]

where the vector \( W \in T_a T^2 M \) projects onto \( v \in T_{\tau_M(z)} M \). This morphism \( \tilde{\delta} f \), called the *Euler–Lagrange morphism* of \( f \), is an affine morphism of affine bundles whose linear part \([\tilde{\delta} f] : V_{\tau_M} \to V^*_{\tau_M} \) is the Hessian map \([10, 19]\). This property is of great importance in our analysis of constrained Lagrangian systems.

### 2.2 Local coordinate expressions of \( T, d_T f, H f \) and \( \tilde{\delta} f \):

A local coordinate system in \( M, (q^i) \), \( i = 1, \ldots, m \), induces natural coordinate systems in \( TM \) and \( TT M \), \( (q^i, v^i) \) and \( (q^i, v^i; a^i, w^i) \) respectively. As submanifolds of \( TT M \), the natural coordinates in \( T^2 M \) and \( V_\tau M \) are \( (q^i, v^i, a^i) \) and \( (q^i, v^i, w^i) \), respectively, with the fibred \( a \)- and \( w \)-coordinates being the same as the tangent \( a \)-coordinates in \( TT M \) but in case of \( u^i = v^i \) and \( u^i = 0 \), respectively.

The vertical bundle \( \pi \) is not a tangent bundle, consequently the sections of its dual bundle \( \pi^* \) cannot be considered as differential forms, although they are closely related; in fact, the local basis \( \partial^j \) of sections of \( \pi^* \), dual of the \( \tau_M \)-vertical vector fields \( \partial / \partial v^i \), satisfy the relations \( \langle \partial^j(z), \xi^V(z, v) \rangle = \langle dq^j(\tau_M(z)), v \rangle \), a property equivalent to their definitions. Notice that for the \( q \)-coordinates in \( T^2 M \) we have \( \delta q^j = -dq^j \), and thereby \( \tilde{\delta} q^j = -\partial^j \circ \mu \).

Taking into account that the local expression of the total time derivative operator \( T \) in the fibre coordinates \( (q, v, a) \) is

\[
T = v^j \left( \frac{\partial}{\partial q^j} \circ \mu \right) + a^j \left( \frac{\partial}{\partial v^j} \circ \mu \right),
\]

the representatives of the total time derivative \( d_T f \) of the function \( f = f(q^i, v^i) \) and its linear part in the local coordinates we have described are as follows:

\[
d_T f = \left( \frac{\partial f}{\partial q^i} \circ \mu \right) v^i + \left( \frac{\partial f}{\partial v^j} \circ \mu \right) a^j, \quad [d_T f] = \left( \frac{\partial f}{\partial v^j} \circ \pi \right) w^j.
\]
Note that in the expressions above the notation makes sense: $T$, a vector field along the projection $\mu$, is expressed as a sum of products of functions of $T^2M$ (the local coordinate functions $v$ and $a$) and (local) vector fields along $\mu$ (namely $(\partial/\partial q) \circ \mu$ and $(\partial/\partial v) \circ \mu$). And $[d_Tf]$ is a sum of products of local functions of $V^\tau_M$.

The first and second fibre derivatives are, respectively, $F_f = \left(\partial f/\partial v_j\right) \circ (dq^j \circ \tau_M)$ and $F^2_f = \left(\partial^2 f/\partial v^j \partial v^k\right) (dq^j \otimes dq^k)$, giving for the Hessian $H_f$ and induced linear morphism $\widehat{H}_f$ the local expressions

$$H_f = \frac{\partial^2 f}{\partial v^j \partial v^k} \vartheta^j \otimes \vartheta^k, \quad \widehat{H}_f = w^j \left(\left(\frac{\partial^2 f}{\partial v^j \partial v^k}\right) \circ \pi\right).$$

Finally, for the Euler–Lagrange form and morphism we have

$$\delta f = \{f\}_j dq^j = -\{f\}_j \delta q^j \quad \text{and} \quad \widetilde{\delta} f = -\{f\}_j \widetilde{\delta} q^j = \{f\}_j (\vartheta^j \circ \mu),$$

where $\{f\}_j$ denotes the (local) function of $T^2M$

$$\{f\}_j = d_T \left(\frac{\partial f}{\partial v^j}\right) - \frac{\partial f}{\partial q^j} \circ \mu.$$  

In the multidimensional case, that is, for a vector-valued function $f \in C^\infty(TM, \mathbb{R}^r)$, with components $(f^1, \ldots, f^r)$ in the canonical basis $\{e_\ell\}_{\ell=1,\ldots,r}$ of $\mathbb{R}^r$, the total-time derivative (1) is a vector-valued function $d_Tf \in C^\infty(T^2M, \mathbb{R}^r)$ with components $(d_Tf_1, \ldots, d_Tf_r)$; its linear part $[d_Tf] \in \mathcal{L}(V^\tau_M, TM \times \mathbb{R}^r)$ and the fibre derivative $\mathcal{F}f : TM \to \mathcal{L}(TM, \mathbb{R}^r) \equiv T^*M \otimes \mathbb{R}^r$ read

$$[d_Tf] = \left(\frac{\partial f^\ell}{\partial v^j} \circ \pi\right) w^j e_\ell, \quad \mathcal{F}f = \frac{\partial f^\ell}{\partial v^j} e_\ell \otimes (dq^j \circ \tau_M).$$

2.3 The Moore–Penrose inverse of a singular linear transformation.

Another very useful tool in our constructions is the ‘inverse of a singular linear transformation’, a concept coming from linear algebra that we have extended to vector and affine bundles and applied to the study of constrained Lagrangian systems in [10]; see also [40]. The concept is based on the universal property of quotient spaces in linear spaces, but there are alternative and equivalent descriptions due to Moore [5] and Penrose [36] giving a quick characterisation more useful in applications.

Let be $F : E \to \bar{E}$ a linear map between Euclidean spaces and $P$ the projector onto the orthogonal complement $(\text{Ker} F)^\perp$; $P$ has the same kernel as $F$. The universal
property of the quotient in a linear space determines a linear isomorphism $\sigma$ between $F(E)$ and $(\text{Ker} F)^\perp$ according to the following commutative diagram:

$$
\begin{array}{cccc}
E & \xrightarrow{P} & E & \xrightarrow{F} \tilde{E} \\
\downarrow{\text{iL}} & & \downarrow{\text{pr}} & \downarrow{\text{iR}} \\
(\text{Ker} F)^\perp & \xrightarrow{\simeq} & E/\text{Ker} F & \xrightarrow{\simeq} F(E) \\
\end{array}
$$

(12)

The linear isomorphism $\sigma$ can be extended to give the linear morphism $F^\dagger : \tilde{E} \to E$, $F^\dagger(v) = \sigma(v)$, when $v \in F(E)$, and $F^\dagger(v) = 0$ when $v \in (F(E))^\perp$. The morphism $F^\dagger$ is known as the Moore–Penrose inverse or simply the inverse of $F$.

Moore’s description of $F^\dagger$ uses orthogonal projectors: the inverse of $F$ is the unique linear map $F^\dagger : \tilde{E} \to E$ such that $F \circ F^\dagger$ is the orthogonal projection onto $F(E)$ and $F^\dagger \circ F$ is the orthogonal projection onto $(\text{Ker} F)^\perp$. Other two useful properties are (i) $F \circ F^\dagger \circ F = F$; (ii) $F^\dagger \circ F \circ F^\dagger = F^\dagger$.

The main application of the inverse $F^\dagger$ is to solve linear equations. In fact, the necessary and sufficient condition for the linear equation $F(v) = \bar{v}$, $\bar{v} \in \tilde{E}$, to have a solution is $\bar{v} \in F(E)$, that is $F(F^\dagger(\bar{v})) = \bar{v}$. Moreover, $v_0 = F^\dagger(\bar{v})$ is the solution of minimal length and any other solution can be written as $v = F^\dagger(\bar{v}) + \text{Ker} F$.

These results can be extended to affine spaces. As far as the differentiable case (linear morphism between vector bundles) is concerned we mention here only that a constant rank morphism $F : E \to \tilde{E}$ between two vector bundles over the base manifold $M$ defines vector sub-bundles Ker $F$ and $F(E)$; when both $E$ and $\tilde{E}$ are Euclidean, $F$ gives rise to an isomorphism $\bar{F}$ of the vector bundles $F(E)$ and $(\text{Ker} F)^\perp$ which can be extended to a morphism $F^\dagger : \tilde{E} \to E$ called the Moore–Penrose inverse of $F$, and enjoys the same properties as those of the purely algebraic case. In particular, the morphisms of orthogonal projection onto $(\text{Ker} F)^\perp$ and $F(E)$ are, respectively,

$$
P_{(\text{Ker} F)^\perp} = F^\dagger \circ F, \quad P_{F(E)} = F \circ F^\dagger.
$$

(13)

The result concerning linear equations, both homogeneous and inhomogeneous, is also valid at the geometric level. For details and more properties, see [10].

3 The vector acceleration and the force in Lagrangian mechanics

Now let us study the motion of a first-order autonomous Lagrangian system, with
Lagrangian function \( L \in C^\infty(TM) \). The Euler–Lagrange equations determine the trajectories as the integral curves of the unique second order differential equation (SODE) field \( \Gamma \in \mathfrak{X}(TM) \) solution of \( i(\Gamma) d\Theta_L + dE_L = 0 \), where \( \Theta_L = S^*dL \) is the Cartan 1-form associated to \( L \) [9].

An equivalent description, the one we adopt throughout the rest of the paper, is based on the fact that a SODE field \( \Gamma \) is equivalent to a section \( \gamma : TM \to T^2M \) of \( \mu \), the relation between them being [10, 11]

\[
\Gamma = T \circ \gamma; \tag{14}
\]

if in local fibered coordinates \( \gamma(q,v) = (q^j, v^j, a^j = \Gamma^j(q,v)) \), it follows from (6) that the equivalent SODE field reads \( \Gamma(q,v) = v^j \partial/\partial q^j + \Gamma^j(q,v) \partial/\partial v^j \). By definition, the integral curves of the section \( \gamma \), i.e. curves \( \sigma : \mathbb{R} \to M \) such that \( \gamma \circ J_1^1 \sigma = J^2 \sigma \), are integral curves of \( \Gamma \), and vice versa. With this equivalence, the dynamical equation can be expressed in terms of the Euler–Lagrange 1-form of \( L \) as \( \gamma^* \delta L = 0 \); taking into account (5), it immediately follows that the trajectories are the integral curves \( \sigma \) of the SODE section \( \gamma \in \text{Sec}(\mu) \) solution of the Euler–Lagrange equation

\[
\widetilde{\delta L} \circ \gamma = 0. \tag{15}
\]

This form of the dynamical equation, written in terms of the Euler–Lagrange morphism of \( L \), \( \widetilde{\delta L} \), is clearly a linear equation in affine bundles, the point of view suitable in our constructions.

For regular Lagrangians, i.e. Lagrangians \( L \) such that \( \widetilde{\mathcal{H}L} \) is a (local) isomorphism, it is well established that the solution of the linear equation (15) is unique. The motion \( \gamma \in \text{Sec}(\mu) \) is completely determined by \( L \). For such Lagrangians, the vertical bundle \( V\tau M \) is endowed with a scalar product structure \( \mathcal{H}L \), which moreover we will assume to be everywhere positive-definite.

Due to the affine character of \( \widetilde{\delta L} \) and recalling that its linear part is the Hessian \( \widetilde{\mathcal{H}L} \), we can write the equation of motion (15) as an equation for the vector deviation \( \gamma_0^{-} \in \text{Sec}(\pi) \) of the actual motion \( \gamma \) from another one \( \gamma_0 \in \text{Sec}(\mu) \) taken as ‘the reference motion’:

\[
\widetilde{\mathcal{H}L} \circ \gamma_0^{-} = F. \tag{16}
\]

In this equation \( \gamma_0^{-} \) is defined by \( \gamma_0^{-}(z) = \gamma_0(z) - \gamma(z), \, z \in TM, \) and we call it the ‘vector acceleration field’ of the \( \gamma \)-motion w.r.t. the reference motion \( \gamma_0 \). The term \( F \) in the right-hand side, defined by

\[
F := \widetilde{\delta L} \circ \gamma_0 \in \text{Sec}(\pi^*), \tag{17}
\]
is called the ‘(Lagrangian) force’ [19]. Thus, the equation of motion (16) can be interpreted as a Newton-like equation ‘mass × acceleration = force’, the ‘mass’ term being the Hessian of the Lagrangian. Any section of μ can be chosen as reference motion γ₀, but in physical theories it should rely on fundamental laws; in classical mechanics, Galilei’s inertia law provides as a universal reference γ₀ the motion in the space of an isolated system of test particles given by (15) for a free Lagrangian (or kinetic energy) L₀, that is, \( \delta \tilde{L}_0 \circ \gamma_0 = 0 \).

The force \( F(z) = -\tilde{\delta}L(\gamma_0(z)) \in V_z^* \tau_M \) acting on the system at the kinematical state \( z \) is given by the Lagrangian \( L \) and the acceleration \( \gamma_0(z) \) of the inertial motion at the same state; the ‘force law’ \( F(z) \) expresses the objective influences on the system which produce the ‘deviation’ (that is, the vector acceleration) \( \gamma_0 \gamma(z) \) at the given state \( z \).

Remark 1. As pointed out above, the force \( F \) cannot be considered as a differential 1-form, though \( F(z) \in V^*_z \tau_M \) may be written in coordinate form as \( F_i dv^i \), but this cotangent vector will in general, when written in a new coordinate system, have terms in both \( dq \) and \( dv \). The right representation is in terms of the basic covectors \( \partial/\partial v^i \), \( F(z) = F_i(z) \partial^i(z) \). Nevertheless, the ‘vertical differential form’ \( F = F_i \partial^i \) is closely related to the \( \tau_M \)-semibasic 1-form \( F_i dq^i \) (see §2.2).

Remark 2. The equation (16) can be written at each state \( z \in TM \) trivially as

\[
D\mathcal{G}_z(\gamma(z)) = F(z),
\]

where \( \mathcal{G}_z \) is the real function on the affine space \( \mu^{-1}(z) \) defined by \( \mathcal{G}_z(a) = (1/2)HL(z) \cdot \langle \gamma_0(z)A, \gamma_0(z)a \rangle \); its differential at the point \( a \) is precisely \( D\mathcal{G}_z(a) = H\ell_z \cdot \gamma_0(z)a \in V_z^* \tau_M \). This fact will be used later on (see section 6).

4 Constrained systems

From now on we focus our attention on the case of a constrained system, that is, a mechanical system described by a regular Lagrangian \( L \) but whose states are restricted to belong to a certain submanifold of \( TM \).

The main problem in constrained mechanics is to find the SODE (i.e., the accelerations in terms of the positions and velocities) of the motion compatible with the applied forces and the constraints.

A constraint submanifold is an imbedded submanifold of the state space, \( C \subset TM \), in which the motion of the system must take place. The constraints express all the independent limitations imposed on the states not accounted for by the definition of
M. They are usually defined by means of a vector-valued function \( f = (f^1, \ldots, f^r) \in C^\infty(TM, \mathbb{R}^r) \), and \( C = f^{-1}(0) \). Analytically the constraints are expressed as a system of \( r \) independent equations \( f^\ell(q, v) = 0, \ell = 1, \ldots, r \), called the ‘constraint equations’. In general, a ‘constraint function’ is any function \( g \in C^\infty(TM) \) vanishing on \( C \).

We are mainly interested in analysing non-holonomic constraints affecting only the velocities, not the configurations, and assume Marle’s regularity condition \([30, 39]\), that is, the restriction to \( C \) of the projection \( \tau_M \) is a submersion \( \pi_C = \tau_M|_C \). This is equivalent to the fact that the fibered derivative \( \mathcal{F}f \) is surjective at each admissible state \( z \in C \) (see \([39]\)). In the fibered coordinates \((q, v)\), which means that the rank of the matrix \( (\partial f^\ell/\partial v^k) \), and that of \( [df_Tf] \), is \( r \) (§2.2).

On the other hand, the constrained motion \( \gamma \) must also satisfy the compatibility condition \( df_Tf \circ \gamma = 0 \), which ensures that, when starting from an admissible initial state \( z \in C \), the constraint itself is fulfilled during the evolution. For this reason, the motions compatible with the constraint are defined on \( C \) and take its values in the submanifold \( D_C = (df_Tf)^{-1}(0) \cap \mu^{-1}(C) \subset T^2M \). The space \( D_z = (df_Tf)^{-1}(0) \cap \mu^{-1}(z) \) is the set of admissible accelerations over the state \( z \in C \), that is, the accelerations compatible with the constraints. It has the structure of an affine subspace of the space of possible accelerations \( \mu^{-1}(z) \) modelled on the vector space \( \text{Ker}_z[df_Tf] \); thus, the total space \( D_C = \cup_{z \in C} D_z \), with the canonical projection over \( C \), is giving a structure of an affine bundle modeled on the restricted bundle \( \text{Ker}_C[df_Tf] \). Consequently, the admissible (i.e. \( C \)-compatible) motions \( \gamma : C \to D_C \) are sections of that affine bundle.

Vectors in the fibre \( \text{Ker}_z[df_Tf] \) are called virtual displacements on \( z \in C \), and they relate two accelerations compatible with the admissible constraints. A ‘field of virtual displacements’ \( \xi \in \text{Sec}(\text{Ker}_C[df_Tf]) \) satisfies the condition \( [df_Tf] \circ \xi = 0 \). Each admissible motion can be expressed as \( \gamma = \tilde{\gamma}_0 + \xi \), where \( \tilde{\gamma}_0 \) is a fixed admissible motion, that is, \( \tilde{\gamma}_0(z) \in D_z, \forall z \in C \), and \( \xi \) is a field of virtual displacements.

An equivalent characterization of the manifolds \( D_C \) and \( \text{Ker}_C[df_Tf] \) is as submanifolds of \( TC \): we have the equivalences \( D_C = T^2M \cap TC \) and \( \text{Ker}_C[df_Tf] = V\pi_C = \text{Ker}(\pi_C^*) \). The first equivalence is a consequence of both the fact that \( T^2M \) is immersed in \( TTM \) by \( T \) and the definition of \( df_Tf \) given in (1). In fact, for \( a \in \mu^{-1}(z), \) \( z \in C \), the condition \( df_Tf(a) = 0 \) means that \( f_{sz} \cdot T(a) = 0 \), that is, \( T(a) \in T_zC \). On the other hand, for a vector \( V \in \text{Ker}_z[df_Tf] \subset V_zT_M, z \in C \), we have \( f_{sz} \cdot V = 0 \) and \( \tau_{Msz} \cdot V = 0 \); consequently, \( V \in T_zC \) and \( (\pi_C)_z \cdot V = 0 \). These characterizations are useful when studying the examples (Section 7).

When constraints are present, the true motion \( \gamma \) is no longer described by the variational equation (15), that is to say \( \delta L \circ \gamma \neq 0 \), and we assume the existence of a section \( Q \in \text{Sec}_C(\pi^*) \) in such a way that the ‘deformed Euler–Lagrange equation’ is
written
\[ \widetilde{\delta L} \circ \gamma = Q. \]  

With this assumption, sometimes called axiom of constraints \[37\], the constraint is removed by introducing the force \( Q \). The affine character of \( \widetilde{\delta L} \) allows us to write \( \widetilde{\delta L} \circ \gamma = \widetilde{\delta L} \circ \gamma_R + \mathcal{H}L \circ \widetilde{\gamma_R} \), where \( \gamma_R \) is the unconstrained or released motion, i.e. the motion solution of equation (15), and \( \widetilde{\gamma_0} \circ \gamma_R = \mathcal{H}L^{-1} \circ F \in \text{Sec}(\pi) \) is the unique field of vector accelerations solution of (16); then we obtain the following equation for the vector acceleration \( \widetilde{\gamma_R} \in \text{Sec}_C(\pi) \) of the \( \gamma \)-motion w.r.t. the released \( \gamma_R \)-motion:
\[ \mathcal{H}L \circ \widetilde{\gamma_R} = Q. \]  

According to (16), \( Q \) can be interpreted as the force separating the \( \gamma \)-motion from the released one \( \gamma_R \), and is called the ‘constraint force’. And taking into account (16) and the affine character of \( \widetilde{\delta L} \) again, the ‘total force’ deviating the \( \gamma \)-motion from the inertial one \( \gamma_0 \) is precisely \( F + Q \), that is
\[ \mathcal{H}L \circ \widetilde{\gamma_0} = F|_C + Q. \]  

Note that the force \( F = -\widetilde{\delta L} \circ \gamma_0 \) is known and defined over all \( TM \) whereas the constraint force \( Q \) is only defined on \( C \) and unknown.

The compatibility condition \( d_T f \circ \gamma = 0 \) can be written in terms of the corresponding vector acceleration \( \widetilde{\gamma_0} \) as
\[ [d_T f] \circ \widetilde{\gamma_0} = b_0|_C, \]  
where \( b_0 \) is defined to be the vector-valued function
\[ b_0 := -d_T f \circ \gamma_0 \in C^\infty(TM, \mathbb{R}^r), \]  
considered as a section of the trivial bundle \( pr_1 : TM \times \mathbb{R}^r \to TM \); \( b_0 \) is determined by the constraint function and the general reference motion \( \gamma_0 \). The equation (22) is the intrinsic formulation of the constraint equation \( f = 0 \) in terms of the accelerations; it is an integrable second-order constraint equation.

Thus the very dynamical problem, once the reference motion \( \gamma_0 \) has been chosen, is to look for the field of vector accelerations \( \widetilde{\gamma_0} \in \text{Sec}_C(\pi) \) solution of both the linear equations (21) and (22). The solution of (22) is expressed by means of the Moore–Penrose inverse of the linear operator \([d_T f]\) with respect to the scalar product structure \( \mathcal{H}L \) in \( V_{\tau M} \) and the standard euclidean product \( g_r \) in \( \mathbb{R}^r \),

\[ \widetilde{\gamma_0} = [d_T f]^{\dagger} \circ b_0|_C + \xi, \]  

where the compatibility condition \( d_T f \circ \gamma = 0 \) can be written in terms of the corresponding vector acceleration \( \widetilde{\gamma_0} \) as
where \( \xi \) is an up to now arbitrary field of virtual displacements. Finally, for the section \( \gamma \) solution of (19) we obtain
\[
\gamma = \gamma_0|_C + [d_Tf]^\dagger \circ b_0|_C + \xi, \quad (25)
\]
from where we can deduce that the constrained motion \( \gamma \) is the addition of a fixed part \( (\gamma_0 + [d_Tf]^\dagger \circ b_0)|_C \) which takes its values in \( D_C \) (thus, it is the fixed admissible motion \( \hat{\gamma}_0 \)), and a vector part \( \xi \), a field of virtual displacements.

Equation (25) gives the general form of the true motion \( \gamma \); recalling the uniqueness of the motion, the remaining problem is to find the unique field of virtual displacements \( \xi \) in such a way that (19) holds, which clearly implies that \( \xi \) depends on the unknown constraint force \( Q \). So, how to find \( Q \)? The answer depends on the character of the constraints according to its action on virtual displacements \( \zeta \), expressed by giving the value of the ‘virtual work’ (or power) \( \langle Q, \zeta \rangle \). The special but important case of ideal constraints is analyzed in the next section.

5 Ideal constraints and the Gauss principle of least constraint

The Gauss principle of least constraint concerns the relation between the constrained motion \( \gamma \) and the unconstrained one \( \gamma_R \) in the case of ideal constraints, that is, constraints whose constraint force does not work in virtual displacements.

Instead of (22) we can write equivalently [10]
\[
[d_Tf] \circ \overrightarrow{\gamma_R} = b_R|_C, \quad (26)
\]
where \( b_R = -d_Tf \circ \gamma_R \in C^\infty(TM, \mathbb{R}^r) \); in fact, we have \( b_0 = b_R + [d_Tf] \circ \overrightarrow{\gamma_0} \).

Now the true motion satisfies both (20) and (26); according to (26) it is of the form \( \overrightarrow{\gamma_R} = [d_Tf]^\dagger \circ b_R|_C + \eta \), with \( \eta \) a field of virtual displacements. Substituting into (20), we obtain that the constraint force \( Q \) is a superposition of two terms:
\[
Q = \overrightarrow{HL} \circ [d_Tf] \circ b_R|_C + \overrightarrow{HL} \circ \eta. \quad (27)
\]
The first term, the ‘ideal part’ of \( Q \), \( Q_{\text{id}} = \overrightarrow{HL} \circ [d_Tf] \circ b_R|_C \), is completely fixed by the constraints and the released motion. Due to the property \( \text{Im}[d_Tf]^\dagger = (\text{Ker}[d_Tf])^\perp \), it is obvious that \( \langle Q_{\text{id}}, \zeta \rangle = 0 \) for an arbitrary virtual displacement \( \zeta \), and in consequence \( Q_{\text{id}} \in \text{Sec}((\text{Ker}_C[d_Tf])^0) \), the superscript “0” denoting the annihilator. The
second term, the ‘non-ideal’ part, \( Q_{\text{id}} = \widehat{HL} \circ \eta \), with \( \eta \in \text{Sec}_C(\text{Ker}[d_T f]) \), is arbitrary and takes its values in the space \( \widehat{HL}([\text{Ker}C[d_T f]]) = ([\text{Ker}C[d_T f]])^\perp_0 \).

From now on we will be concerned with a system subjected only to ideal constraints: \( Q_{\text{id}} = 0 \) and therefore \( \eta = 0 \); in this case it is easy to find that
\[
\overrightarrow{\gamma}_R = [d_T f]^\dagger \circ b_R|_C,
\]
and, using (21), the acceleration \( \overrightarrow{\gamma}_0 \) turns out to be
\[
\overrightarrow{\gamma}_0 = \widehat{HL}^{-1} \circ F + [d_T f]^\dagger \circ b_R \quad (\text{on } C).
\]
This result solves the main problem (21)–(22) for ideal constraints; for more details about this approach, see [10].

We remark that deviation (28) is the solution of the linear equation (26) having minimal length—a property of the Moore–Penrose inverse (Section 2.3)—, a fact which has an interesting geometrical interpretation, due to Gauss. The length of a vector \( w \in V_{\tau_M} \) is defined by the scalar product structure \( HL, ||w|| = (HL_{\pi(w)}(w, w))^{1/2} \).

Had we assumed that \( HL \) is positive-definite at each point \( z \), the vector \( w = \overrightarrow{\gamma}_R(z) \in V_{z,\tau_M} \) solution of (26) of minimal length is also a minimum of the quadratic form
\[
G_z(w) := \frac{1}{2} HL_z(w, w), \quad w \in V_{z,\tau_M},
\]
restricted to the solution space \( A_z, z \in C \), of the linear equation (26), \( A_z := \{ w \in V_{z,\tau_M} \mid [d_T f]_z \cdot w = b_R(z) \} \). \( A_z \) is an affine sub-space of \( V_{z,\tau_M} \) with associated vector space the vector sub-space \( \text{Ker}_z[d_T f] \) of virtual displacements over the state \( z \). Then \( T_w A_z = \text{Ker}_z[d_T f] \) and the necessary condition for a minimum is \( DG_z(w)|_{\text{Ker}_z[d_T f]} = 0 \).

The quadratic real functions \( G_z \) (30) are defined on the fibres of \( \pi \), but taken all together they define the function \( G \in C^\infty(V_{\tau_M}) \) by means of \( G(w) = G_{\pi(w)}(w) \), called the ‘Gauss constraint’. Thus, we can say that the actual motion \( \gamma \) is such that its deviation from the released one \( \gamma_R \) at each admissible state \( z \in C \) is a minimum of the Gauss constraint \( G \) restricted to the subspace \( A_z \), a property of the motion known as the Gauss principle of least constraint (see e.g. [34, 41]), a very minimum principle that can be put at the foundations of mechanics (with ideal constraints).

To end this section, it is worthy noting that the decomposition \( \bar{Q} = Q_{\text{id}} + Q_{\text{nid}} \) pointed out above for the constraint force \( Q \) (27) is quite natural, because the splitting \( V_{\tau_M} = \text{Ker}[d_T f] \oplus ([\text{Ker}[d_T f]] \perp ^\perp \) gives the dual splitting \( V^*_{\tau_M} = \widehat{HL}(V_{\tau_M}) = ([\text{Ker}[d_T f]] \perp ^\perp) \oplus ([\text{Ker}[d_T f]] \perp ^\perp) \). Recalling that \( P_{([\text{Ker}[d_T f]])} = [d_T f]^\dagger \circ [d_T f] \) (Section 2.3),
the corresponding projectors are

\[ P_{(Ker[d_T f])^0} = \hat{H}_L \circ P_{(Ker[d_T f])^0} \circ \hat{H}_L^{-1} = \hat{H}_L \circ [d_T f]^\dagger \circ [d_T f] \circ \hat{H}_L^{-1}, \]
\[ P_{(Ker[d_T f])^1} = \hat{H}_L \circ P_{Ker[d_T f]} \circ \hat{H}_L^{-1} = \hat{H}_L \circ (\mathbb{I} - [d_T f]^\dagger \circ [d_T f]) \circ \hat{H}_L^{-1}, \]  

(31)

with \( \mathbb{I} = id_{V_T M} \). Of course, the force \( F \) is also decomposed the same way, \( F = F_{id} + F_{nid} \), with

\[ F_{id} = \hat{H}_L \circ [d_T f]^\dagger \circ [d_T f] \circ \hat{H}_L^{-1} \circ F, \]
\[ F_{nid} = \hat{H}_L \circ (\mathbb{I} - [d_T f]^\dagger \circ [d_T f]) \circ \hat{H}_L^{-1} \circ F. \]

(32)

The ideal part does annihilate virtual displacements \( \eta \), \( \langle F_{id}, \eta \rangle = 0 \), while the non-ideal part does not, \( \langle F_{nid}, \eta \rangle \neq 0 \). This decomposition is unique and the isomorphism \( (\mathbb{I} - [d_T f]^\dagger \circ [d_T f]) \) means that \( F_{nid} \in \text{Sec}((\mathbb{I} - [d_T f]^\dagger \circ [d_T f]))^* \).

### 6 The Gibbs–Appell equations

Assuming the Gauss principle as the starting point, and in order to get more important and useful properties of the constrained motion, it is interesting to look at the dynamical problem from a new perspective: given the unconstrained motion \( \gamma_R \), solution of (16), the acceleration \( \gamma(z) \) of the constrained motion at the state \( z \) is such that the variation of the Gauss constraint \( \mathcal{G}_z \) at the point \( w = \gamma_R(z) \) vanishes on virtual displacements, that is

\[ DG_z(w)|_{\mathcal{G}_z} = 0, \]

(33)

as we have pointed out before. Now we want to show that, as is already known, when this condition is written in terms of the acceleration \( \gamma(z) \in D_z \) itself, we get the equation derived independently by Gibbs (in 1879) and Appell (in 1899), see [1] and [21], respectively.

Let \( \gamma \) be the true motion; let \( \gamma_R \) and \( \gamma_0 \) be the corresponding unconstrained motion and the inertial one, respectively. The Gauss constraint \( \mathcal{G}_z \) (30) allows us to define a real function \( \mathcal{G}_z \in C^\infty(\mu^{-1}(z)) \) by \( \mathcal{G}_z(a) = \mathcal{G}_z(\gamma_0(z)a) \), \( a \in \mu^{-1}(z) \), and by extension a real function \( \mathcal{G} \in C^\infty(T^2M) \) in such a way that \( \mathcal{G}(a) = \mathcal{G}_\mu(a) \); i.e.

\[ \mathcal{G}(a) := \frac{1}{2} \mathcal{H}_L(a, a), \]

(34)
with \( z = \mu(a) \). It measures the \( \mathcal{H}L \)-distance between the given acceleration \( a \) and the acceleration of the inertial motion \( \gamma_0 \) at the same state \( z \); when restricted to the space \( D_C \) of admissible accelerations, \( \mathcal{G} \) is denoted by \( \mathcal{G}^0 \) and called the ‘Gibbs function’.

By definition, for \( a \in \mu^{-1}(z) \) and \( w \in V_z \mathcal{T}_M \),
\[
\mathcal{G}_z(a + w) = \mathcal{G}_z(a) + \mathcal{H}L_z(\overrightarrow{\gamma_0(z)a}, w) + \mathcal{G}_z(w). \tag{35}
\]
Taking into account that \( \mathcal{G}_z \) is quadratic, the differential of \( \mathcal{G}_z \) at the point \( a \) is
\[
\text{D}G_z(a) = \mathcal{H}L_z \cdot \overrightarrow{\gamma_0(z)a}. \tag{36}
\]
Therefore we obtain the following relation between the differentials of \( \mathcal{G}_z \) and \( \mathcal{G}_z \) at the points \( a \in \mu^{-1}(z) \) and \( w \in V_z \mathcal{T}_M \), respectively:
\[
\text{D}G_z(a + w) = \text{D}G_z(a) + \text{D}G_z(w). \tag{37}
\]
This means that the differential of \( \mathcal{G}_z \), \( \text{D}G_z : \mu^{-1}(z) \to V_z^* \mathcal{T}_M \), is an affine application with a linear part the differential of the Gauss constraint, \( \text{D}G_z : V_z \mathcal{T}_M \to V_z^* \mathcal{T}_M \). The same relation holds for their restrictions \( \mathcal{G}_z^0 \) and \( \mathcal{G}_z^0 \) to \( D_z \) and \( \text{Ker}_z[d_T f] \), respectively, with \( a \in D_z \) and \( w \in \text{Ker}_z[d_T f] \), a virtual displacement.

The necessary condition for \( a = \gamma(z) \) to be the acceleration of the true motion at the dynamical state \( z \in C \) is, as stated by the Gauss principle, \( \text{D}G_z(\overrightarrow{\gamma_R(z)}a)|_{\text{Ker}_z[d_T f]} = 0 \); putting \( a = \gamma_R(z) \) and \( w = \overrightarrow{\gamma_R(z)} \) in (37) and recalling that \( \text{D}G_z(\overrightarrow{\gamma_R(z)}) = \mathcal{H}L_z \cdot \overrightarrow{\gamma_0 \gamma_R(z)} = F(z) \) (see (36) and (16)), this condition can be expressed in the form
\[
\text{D}G_z(\gamma(z)) = F(z), \text{ on Ker}_z[d_T f], \tag{38}
\]
or, in terms of the Gibbs function \( \mathcal{G}_z^0 \in C^\infty(D_z) \) and the non-ideal part of the Lagrangian force \( F \) (32):
\[
\text{D}G_z^0(\gamma(z)) = F_{\text{nid}}(z). \tag{39}
\]

Equation (39) is nothing but the famous Gibbs–Appell equations. It determines the acceleration \( \gamma(z) \) of the true motion. This important result can be put in a compact and global form. The derivatives \( \text{D}G_z^0 \) just form the fibre derivative of the Gibbs function, \( \mathcal{FG}_z^0 : D_z \to (\text{Ker}_C[d_T f])^* \), in such a way that \( \mathcal{FG}_z^0(a) = \text{D}G_z^0(a) \in \text{Ker}_z^*[d_T f], \ a \in D_z \). In conclusion, the solution \( \gamma \) of the dynamical problem is the solution of the linear equation
\[
\mathcal{FG}_z^0 \circ \gamma = F_{\text{nid}}|_{\text{C}}. \tag{40}
\]
This is the global form of the Gibbs–Appell equations. Recall that this is only the necessary condition for \( \gamma \) to be the constrained motion of the system, but assuming that \( \mathcal{H}_L \) is positive-definite, the condition is sufficient too. The linear problem (40) is reflected in the following commutative diagram:

\[
\begin{array}{ccc}
D & \xrightarrow{FG^0} & (\text{Ker}_{C[d_Tf]})^* \\
\gamma & \uparrow & \\
C & \xrightarrow{F_{\text{nid}}} & 
\end{array}
\]

(41)

Note how only the non-ideal part of the force \( F \) appears in (41), while the ideal part is lost; we call \( F_{\text{nid}} \) the effective force.

If in the previous derivation of the Gibbs–Appell equations we use the general expression (25) for the true acceleration, we obtain that the Gibbs–Appell equations can be written for the undetermined virtual displacement \( \xi \) giving the motion (25). In fact, putting \( a = \gamma_0(z) + [d_Tf]^\dagger \cdot b_0(z) \) and \( w = \xi(z) \) in the general relation (37), we have \( D_G z(\gamma(z)) = D_G z(\gamma_0(z) + [d_Tf]^\dagger \cdot b_0(z)) + D_G z(\xi(z)) \); but by (36) \( D_G z(\gamma_0(z) + [d_Tf]^\dagger \cdot b_0(z)) = \mathcal{H}_L z \cdot ([d_Tf]^\dagger \cdot b_0(z)) \) and this is null when restricted to the space of virtual displacements \( \text{Ker}_z[d_Tf] \) because \( \text{Im}[d_Tf]^\dagger = (\text{Ker}_z[d_Tf])^\perp \) (see section 2.3). It immediately follows that if \( \gamma(z) \) satisfies the Gibbs–Appell equations (39), then \( \xi(z) \) satisfies an equation of the same kind:

\[
D_G^0(z(\xi(z))) = F_{\text{nid}}(z),
\]

(42)

and conversely. By extension we also have

\[
FG^0 \circ \xi = F_{\text{nid}}|_C,
\]

(43)

with \( FG^0 \) being the fibre derivative of the restriction \( G^0 \) of the Gauss constraint \( G \) to the space of virtual displacements \( \text{Ker}_C[d_Tf] \). A direct calculation shows that \( FG^0 = \mathcal{H}_L|_{\text{Ker}_C[d_Tf]} \).

Once (43) has been solved for \( \xi \), the solution of the mechanical constrained problem is then (25), that is, \( \gamma = (\gamma_0 + [d_Tf]^\dagger \circ b_0)|_C + \xi \).

**Remark.** The Gibbs–Appell equations (40) or (43) can be considered as the general common form of the equations of the motion in Lagrangian mechanics, for even in the unconstrained case, the law of motion can be put in this form as we have mentioned in remark 2 in section 3, with the Gibbs function defined over all \( T^2M \) and the effective force being nothing but the full applied force \( F \).
Moreover, it can be extended immediately to the case of having a system subjected to a given force \( F \) not included in the Lagrangian, that is, we have a free Lagrangian \( L_0 \), which determines the Hessian and the Gibbs function, and a section \( F \in \text{Sec}(\pi^*) \) giving the force. See the example 5 below.

7 The Gibbs–Appell equations in local coordinates. Examples

Now let us analyse how the Gibbs–Appell equations, in both the unconstrained (18) and the constrained (40) cases, look like in several types of systems of local coordinates, including the so called ‘quasi-velocities’ as coordinates for the velocities.

1. First of all we will consider the simple case when the constraints are absent. Let \( L = L_0 + L_{\text{int}} \in C^\infty(TM) \) be a Lagrangian of mechanical type, where \( L_0 \) is a free Lagrangian (or kinetic energy) and \( L_{\text{int}} \) is an interaction Lagrangian which is at most linear in the velocities. \( L_0 \) represents a wider class of mechanical systems such as a ‘system of free particles’ or a ‘free rigid body’ determining the inertial (or free) motion \( \gamma_0 \) as the solution of the linear homogeneous equation \( \delta L_0 \circ \gamma_0 = 0 \). And \( L_{\text{int}} \) gives the external force \( F \in \text{Sec}(\pi^*) \) defined by the Lagrangian \( L \) because

\[
F = -\tilde{\delta}L \circ \gamma_0 = -\tilde{\delta}L_{\text{int}} \circ \gamma_0.
\]

In the natural local coordinates described in section 2.2 we have (see (9))

\[
\tilde{\delta}L_0 = \{L_0\}_i(\vartheta^i \circ \mu) \quad \text{and} \quad \tilde{\delta}L_{\text{int}} = \{L_{\text{int}}\}_i(\vartheta^i \circ \mu),
\]

where \( \vartheta^i \) are the covectors dual of the basic vertical vectors \( \partial/\partial v^i \) (see section 2.2) and the local functions \( \{L_0\}_i, \{L_{\text{int}}\}_i \in C^\infty(T^2M) \) are given by (10). If the local expression of the inertial motion \( \gamma_0 \) in these coordinates is \( \gamma_0(q,v) = (q,v,\Gamma_0(q,v)) \), the components \( \Gamma_0^i \) of the free acceleration are the solution of the linear system

\[
\frac{\partial^2 L_0}{\partial v^i \partial v^j} \Gamma_0^j = - \frac{\partial^2 L_0}{\partial q^j \partial v^i} v^j + \frac{\partial L_0}{\partial q^i}; \quad (44)
\]

the components \( F_i \) of the force \( F = F_i \vartheta^i \) will be

\[
F_i(q,v) = - \frac{\partial^2 L_{\text{int}}}{\partial q^j \partial v^i} v^j + \frac{\partial L_{\text{int}}}{\partial q^i}. \quad (45)
\]
In order to find the dynamics \( \gamma(q, v) = (q, v, a = \Gamma(q, v)) \), or equivalently the vector acceleration \( \gamma_0^\gamma(q, v) = (q, v, w = \Gamma(q, v) - \Gamma_0(q, v)) \), we have to construct the Gibbs function \( G \) and then solve the Gibbs–Appell equations \( F G(\gamma(q, v)) = F(q, v) \) (40). The Hessian \( HL \) is (see (8))

\[
HL = g_{ij} \partial^i \otimes \partial^j, \quad \text{with} \quad g_{ij} = \frac{\partial^2 L_0}{\partial v^i \partial v^j},
\]

and then in this case the Gauss constraint is \( G(q, v, w) = g_{ij} w^i w^j / 2 \), which in turn gives for the Gibbs function (34)

\[
G(q, v; a) = \frac{1}{2} g_{ij}(\gamma(q, v)) (a^i - \Gamma_0^i) (a^j - \Gamma_0^j),
\]

which gives rise to the Gibbs–Appell equations (18) \( (\partial G / \partial a^i) \gamma(\gamma(q, v)) = F_i(q, v) \partial \gamma(q, v)^i \), namely

\[
g_{ij}(\Gamma^j - \Gamma_0^j) = F_i.
\]

But taking into account (44) and (45), we see that equations (47) lead to the conventional Euler–Lagrange ones for the accelerations \( \Gamma^i \):

\[
\frac{\partial^2 L}{\partial v^i \partial v^j} \Gamma^j + \frac{\partial^2 L}{\partial q^j \partial v^i} v^j - \frac{\partial L}{\partial q^i} = 0,
\]

where \( L = L_0 + L_{\text{int}} \).

In the case of \( L \) being a general Lagrangian, in non-mechanical theories, we have to select a section \( \gamma_0 \) as the ‘inertial’ motion according to the corresponding theory.

2. Analysis with quasi-velocities: We can replace the velocities \( v \) by other coordinates \( \nu \), called quasi-velocities, which are linearly related with the \( v \)-velocities, i.e. \( v^i = A^i_j(q) \nu^j \), where the matrix \( A = (A^i_j) \) is invertible, and then \( \nu^\ell = B^\ell_k(q) v^k \), with \( A^i_j B^j_\ell = \delta^i_\ell \). See [12] to better understand the geometrical origin of a system of quasi-velocities; see also [7]. The new coordinates are now: \((q, \nu, a = \dot{\nu}) \) in \( TTM \) (here \( \alpha \) is the ‘velocity’ of the quasi-velocity \( \nu \)), \((q, \nu, \alpha) \) in \( T^2M \) (where \( u = v \)) and \((q, \nu, \omega) \) in \( V_\tau M \) (where \( u = 0 \)); the corresponding changes of coordinates are given by

- in \( T M \) : \((q^i, \nu^i) \mapsto (q^i, \nu^i = A^i_i \nu^i); \)
- in \( TTM \) : \((q, \nu; u, \alpha) \mapsto (q^i, v^i = A^i_i \nu^i; u^i, a^i = A^i_i \alpha^i + \frac{\partial A^i_k}{\partial q^k} u^k \nu^i); \)
- in \( T^2M \) : \((q, \nu, \alpha) \mapsto (q^i, v^i = A^i_i \nu^i; a^i = A^i_i \alpha^i + \frac{\partial A^i_k}{\partial q^k} A^k_m v^m \nu^i); \)
- in \( V_\tau M \) : \((q, \nu, \omega) \mapsto (q^i, \nu^i = A^i_i \nu^i; w^i = A^i_i \omega^i). \)
According to the change of coordinates in \( V_T \) the relation between the covectors \( \vartheta^\nu \) dual of the basic vertical vectors \( \partial/\partial \nu \) and the covectors \( \vartheta^i \) is \( \vartheta^i = A^i_\ell \vartheta^\nu \); consequently the local coordinate expression of the force \( F \) is given by \( F(q, \nu) = F_i(q, \nu) \vartheta^i = \Phi_\ell(q, \nu) \vartheta^\nu \), that is, the components of \( F \) in the new basis are \( \Phi_\ell = A^i_\ell F_i \).

In the new variables, the Hessian \( \mathcal{H}L \) (46) is

\[
\mathcal{H}L = g_{mn} \vartheta^m \otimes \vartheta^n,
\]

where \( g_{mn} = A^i_m A^j_n g_{ij} = \partial^2 L_0/\partial \nu^m \partial \nu^n \), with \( L_0(q, \nu) = L_0(q, \nu = A\nu) \). If the inertial dynamics is given by \( \gamma_0(q, \nu) = (q, \nu, \alpha = \Gamma_0(q, \nu)) \), then Gibbs’ function in terms of \((q, \nu, \alpha)\)

\[
G(q, \nu; \alpha) = \frac{1}{2} g_{mn}(q, \nu) (\alpha^m - \Gamma_0^m)(\alpha^n - \Gamma_0^n).
\]

The Gibbs–Appell equations \( (\partial G/\partial \alpha^m)(\gamma) = \Phi_m \) for the dynamics \( \gamma(q, \nu) = (q, \nu, \alpha = \Gamma(q, \nu)) \) turn out to be

\[
g_{mn}(\Gamma^n - \Gamma_0^n) = \Phi_m,
\]

namely \( \Gamma^n = \Gamma_0^n + g^{nm} \Phi_m \), with \( g_{tm} g^{mn} = \delta^m_t \). Note that the equations (47) and (48) are equal in form.

We can write equations (48) in a more explicit way in terms of the configuration coordinates \( q \) and the quasi-velocities \( \nu \) and obtain the so called Hamel–Boltzmann equations

\[
\frac{\partial^2 L_0}{\partial \nu^m \partial \nu^n} \Gamma^n + \frac{\partial^2 L_0}{\partial q^k \partial \nu^m} A^k_\ell \nu^\ell - h^i_{m\ell} \frac{\partial L_0}{\partial \nu^\ell} - A^i_m \frac{\partial L_0}{\partial q^\ell} = \Phi_m,
\]

where

\[
h^i_{m\ell} = A^i_m A^k_\ell \left( \frac{\partial B^i_j}{\partial q^k} - \frac{\partial B^i_k}{\partial q^j} \right)
\]

are said to be the Hamel symbols associated with the quasi-velocities \( \nu \), see [12].

3. A particular example: plane motion of a unity mass particle in polar coordinates \((r, \varphi), \; r \neq 0\). In the natural fibered coordinates \( v_r, v_\varphi \) for velocities, the Lagrangian is

\[
L = L_0 - V(r, \varphi) = \frac{1}{2}(v^2_r + r^2 v^2_\varphi) - V(r, \varphi).
\]

As \( \tilde{\Delta}L_0 = (a_r - rv^2_\varphi)(\partial_\varphi \circ \mu) + (r^2a_\varphi + 2rv_r v_\varphi)(\partial_\nu \circ \mu) \), the accelerations of the inertial motion \( \gamma_0(r, \varphi, v_r, v_\varphi) = (r, \varphi, v_r, v_\varphi, \Gamma^0_r, \Gamma^0_\varphi) \) are

\[
\Gamma^0_r = rv^2_\varphi \quad \text{and} \quad \Gamma^0_\varphi = -\frac{2v_r v_\varphi}{r}.
\]
The Euler–Lagrange form of \( L \) is 
\[
\widetilde{L} = \widetilde{L}_0 - (\partial V/\partial r)(\vartheta_r \circ \mu) - (\partial V/\partial \varphi)(\vartheta_\varphi \circ \mu)
\]
and therefore the force is given by 
\[
F = -(\partial V/\partial r)\vartheta_r - (\partial V/\partial \varphi)\vartheta_\varphi.
\]
Moreover, as the Hessian is represented by 
\[
\mathcal{H}L = \vartheta_r \otimes \vartheta_r + r^2 \vartheta_\varphi \otimes \vartheta_\varphi,
\]
the Gauss constraint (30) is 
\[
\mathcal{G} = \left( w_1^2 + r^2 w_\varphi^2 \right)/2
\]
and therefore the force is given by 
\[
\gamma \text{ and then the dynamics are such that } \Gamma_r = -\partial V/\partial r \text{ and } r^2(\Gamma_\varphi + 2v_r v_\varphi/r) = -\partial V/\partial \varphi.
\]
In other words, the accelerations of the dynamics are 
\[
\Gamma_r = -\frac{\partial V}{\partial r} + rv_\varphi^2, \quad \Gamma_\varphi = -\frac{1}{r^2} \frac{\partial V}{\partial \varphi} - \frac{2v_r v_\varphi}{r}
\]
yielding the equations of motion 
\[
\dot{r} = v_r, \quad \dot{\varphi} = v_\varphi, \quad \dot{v}_r = \Gamma_r, \quad \dot{v}_\varphi = \Gamma_\varphi.
\]
We can also deal with the same problem but in terms of the quasi-velocities \( \nu^1, \nu^2 \) defined by \( \nu^1 = v_r, \nu^2 = r^2 v_\varphi \), namely, \( v_r = \nu^1 \) and \( v_\varphi = \nu^2/r^2 \). The ‘accelerations’ \( \alpha = \dot{\nu} \) in \( T^2M \) are related with natural accelerations \( a \) by means of \( \alpha^1 = a_r \) and \( \alpha^2 = r^2 a_\varphi + 2r v_r v_\varphi \), while the components of the vector acceleration (in \( V_T M \)) are 
\[
\omega^1 = w_r, \quad \omega^2 = r^2 w_\varphi.
\]
Using all these changes of coordinates, we have:
\[
\gamma_0(r, \varphi, \nu^1, \nu^2) = \left( r, \varphi, \nu^1, \nu^2, \Gamma_0^1 = \frac{(\nu^2)^2}{r^3}, \Gamma_0^2 = 0 \right),
\]
\[
F(r, \varphi, \nu^1, \nu^2) = -\frac{\partial V}{\partial r} \vartheta_{\nu^1} - \frac{1}{r^2} \frac{\partial V}{\partial \varphi} \vartheta_{\nu^2},
\]
\[
\mathcal{H}L(r, \varphi, \nu^1, \nu^2) = \vartheta_{\nu^1} \otimes \vartheta_{\nu^1} + \frac{1}{r^2} \vartheta_{\nu^2} \otimes \vartheta_{\nu^2},
\]
\[
\mathcal{G}(r, \varphi, \nu^1, \nu^2, \omega^1, \omega^2) = \frac{1}{2} \left( (\omega^1)^2 + \frac{1}{r^2} (\omega^2)^2 \right),
\]
\[
\mathcal{G}(r, \varphi, \nu^1, \nu^2, \alpha^1, \alpha^2) = \frac{1}{2} \left( (\alpha^1 - \Gamma_0^1)^2 + \frac{1}{r^2} (\alpha^2 - \Gamma_0^2)^2 \right),
\]
and then the dynamics \( \gamma(r, \varphi, \nu^1, \nu^2) = (r, \varphi, \nu^1, \nu^2, \Gamma^1, \Gamma^2) \) is given by Gibbs–Appell equations \( (\partial \mathcal{G}/\partial \alpha^i)|_{\gamma} = \Phi_i \) which give rise to 
\[
\Gamma^1(r, \varphi, \nu^1, \nu^2) = -\frac{\partial V}{\partial r} + \frac{(\nu^2)^2}{r^3}, \quad \Gamma^2(r, \varphi, \nu^1, \nu^2) = -\frac{\partial V}{\partial \varphi}.
\]
Thus, the differential equations of the motion in the state variables \( r, \varphi, \nu^1, \nu^2 \) are 
\[
\dot{r} = \nu^1, \quad \dot{\varphi} = \nu^2/r^2, \quad \dot{\nu}^1 = \Gamma^1, \quad \dot{\nu}^2 = \Gamma^2.
\]
4. Constrained systems: suppose that there are \( r \) independent constraint relationships which do depend on velocities, i.e. \( f^\ell(q,v) = 0, \ell = 1, \ldots, r \), and such that \( \text{rank}(\partial f^\ell/\partial v^k) = r \); thus, they do not affect to base coordinates \( q \) and determine \( n = m - r \) independent velocities \( \nu^1, \nu^2, \ldots, \nu^n \). On the manifold of constraints \( C = f^{-1}(0) \subset TM \) we have a system of independent coordinates \( (q^i, \nu^d) \) and in this way we have some relations of the following type:

\[
v^i = \varphi^i(q, \nu),
\]

in such a way that \( f^\ell(q, \varphi(q, \nu)) = 0 \).

Let us introduce local coordinates in the manifolds \( C \) and \( D_C = T^2M \cap TC \) (section 4). If \( q = (q^1, \ldots, q^m) \) are local coordinates in \( M \), we can consider the tangent bundle coordinates \( (q,v) \) in \( TM \) and the constraint submanifold \( C \) is given by local coordinates \( C = \{(q, \nu)\} = \{(q, v) \in TM \mid v = \varphi(q, \nu)\} \). Its tangent bundle \( TC \) is given locally by

\[
TC = \{(q, \nu; u, \alpha)\} = \{(q, v; u, a) \in TT_M \mid v = \varphi(q, \nu), a = \frac{\partial \varphi}{\partial q} u + \frac{\partial \varphi}{\partial \nu} \alpha\},
\]

and therefore we have the following representation for \( D_C \) (where \( u = v \)):

\[
D_C = \{(q, \nu, \alpha)\} = \{(q, v, a) \in T^2M \mid v = \varphi(q, \nu), a = \frac{\partial \varphi}{\partial q} \varphi + \frac{\partial \varphi}{\partial \nu} \alpha\}.
\]

In a similar way, the set of virtual displacements \( \text{Ker}_C[d_{Tf}] = V\pi_C \) is described with local coordinates \( (q, \nu, \omega) \) (taking \( u = 0, \omega = \alpha \)),

\[
V\pi_C = \{(q, \nu, \omega)\} = \{(q, v, w) \in V\tau_M \mid v = \varphi(q, \nu), w = \frac{\partial \varphi}{\partial \nu} \omega\}.
\]

In order to write the Gibbs–Appell equations in coordinates \( (q, \nu) \) we should first express the Gibbs function \( G^0 \) in terms of \( (q, \nu, \alpha) \) and compute the effective force \( F_{\text{fid}}|_C = \Phi_d \partial^d \) in terms of the independent variables \( (q, \nu) \). The fiber derivative of \( G^0 \) will be \( \mathcal{F}G^0 = (\partial G^0/\partial \alpha^d) \partial^d \), and consequently the linear independence of the covectors \( \partial^d \) yields the Gibbs–Appell equations for the true motion \( \gamma(q, \nu) = (q, \nu, \alpha = \Gamma(q, \nu)) \)

\[
\left. \frac{\partial G^0}{\partial \alpha^d} \right|_\gamma = \Phi_d, \ d = 1, \ldots, n.
\]

Instead of doing it in a generic case we specialize to two specific cases: an example with a linear constraint proposed by Lewis [29] and a particular case of the so called Appell particle, an important example because the constraint is non-linear.
5. Let us consider the example in $\mathbb{R}^3$ studied by Lewis in [29]; in Cartesian coordinates $(x, y, z)$ it is described by the kinetic term $L_0 = (v_x^2 + v_y^2 + v_z^2)/2$ (mass unit) and a generic force $F = F_x \dot{x} + \cdots$ subject to the linear constraint $f = v_z - yv_x = 0$. If we let $\nu^1 = v_y$ and $\nu^2 = (1 + y^2)v_x$ be the quasi-velocities, then $C$ is described by the variables $(x, y, z, \nu^1, \nu^2)$, namely

$$
C = \{(x, y, z, \nu^1, \nu^2) \} = \left\{(x, y, z, v_x = \frac{\nu^2}{1 + y^2}, v_y = \nu^1, v_z = \frac{y \nu^2}{1 + y^2}) \in T \mathbb{R}^3 \right\}.
$$

In the tangent bundle $TC$ the coordinates are $(x, y, z, \nu^1, \nu^2; u_x, u_y, u_z, \alpha^1, \alpha^2)$ and the transformation to velocities and accelerations of $TT \mathbb{R}^3$

$$
v_x = \frac{\nu^2}{1 + y^2}, v_y = \nu^1, v_z = \frac{y \nu^2}{1 + y^2};
a_x = \frac{\alpha^2}{1 + y^2} - \frac{2y u_y \nu^2}{(1 + y^2)^2}, a_y = \alpha^1, a_z = \frac{y \alpha^2}{1 + y^2} + \frac{(1 - y^2)u_y \nu^2}{(1 + y^2)^2}.
\tag{50}
$$

In a similar way, the coordinates in $D_C$ are $(x, y, z, \nu^1, \nu^2, \alpha^1, \alpha^2)$, and the transformation equations are obtained from (50) putting $u = v$:

$$
v_x = \frac{\nu^2}{1 + y^2}, v_y = \nu^1, v_z = \frac{y \nu^2}{1 + y^2};
a_x = \frac{\alpha^2}{1 + y^2} - \frac{2y \nu^1 \nu^2}{(1 + y^2)^2}, a_y = \alpha^1, a_z = \frac{y \alpha^2}{1 + y^2} + \frac{(1 - y^2)\nu^1 \nu^2}{(1 + y^2)^2}.
\tag{51}
$$

In $\ker_C [d_T f] = V \pi_C$ the coordinates are $(x, y, z, \nu^1, \nu^2, \omega^1, \omega^2)$ with transformation equations obtained from (50) putting $u = 0, a = w$ and $\alpha = \omega$; that is

$$
w_x = \frac{\omega^2}{1 + y^2}, w_y = \omega^1, w_z = \frac{y \omega^2}{1 + y^2}.
\tag{52}
$$

The total time derivative of the constraint function $f$ is $d_T f = -ya_x + a_z - v_x v_y$ and its linear part is $[d_T f] = -yw_x + w_z$. As the Hessian of the Lagrangian is $\mathcal{H}L_0 = \partial_x \otimes \partial_x + \cdots$, the local representative of the inverse $[d_T f]^\dagger$ is

$$
[d_T f]^\dagger = \frac{1}{1 + y^2} \left( -y \frac{\partial}{\partial v_x} + \frac{\partial}{\partial v_z} \right);
$$

while the Gauss constraint is $G = (w_x^2 + w_y^2 + w_z^2)/2$. Then the non-ideal part of the given force $F$ (32) is

$$
F_{\text{nid}} = \frac{F_x + y F_z}{1 + y^2} \dot{x} + F_y \dot{y} + \frac{y (F_x + y F_z)}{1 + y^2} \dot{z},
$$

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whose restriction to $C$, i.e. the effective force, is

$$F_{\text{nid}}|_C = F_y\nu^1 + \frac{F_x + yF_z}{1 + y^2}\nu^2.$$  (53)

On the other hand, as the inertial dynamics is null $\gamma_0(x,v) = (x,v,a = 0)$, the Gibbs function in the independent variables $(x,y,z,\nu^1,\nu^2,\alpha^1,\alpha^2)$ is, using (51),

$$G^0 = \frac{1}{2}\left[\left(\frac{\alpha^2}{1 + y^2} - \frac{2y\nu^1\nu^2}{(1 + y^2)^2}\right)^2 + (\nu^3)^2 + \left(\frac{y\alpha^2}{1 + y^2} + \frac{(1 - y^2)\nu^1\nu^2}{(1 + y^2)^2}\right)^2\right].$$

Consequently the Gibbs–Appell equations $[\partial G^0/\partial \alpha^d](\gamma)\vartheta^d = F_{\text{nid}}|_C$ for the true motion $\gamma(x,y,z,\nu^1,\nu^2) = (x,y,\nu^1,\nu^2,\Gamma^1,\Gamma^2)$ give the constrained dynamics

$$\Gamma^1 = F_y, \quad \Gamma^2 = F_x + yF_z + \frac{y\nu^1\nu^2}{1 + y^2}.$$  (54)

The differential equations of the motion in the variables $(x,y,z,\nu^1,\nu^2)$ are obtained putting $\dot{\nu}^d = \Gamma^d$ together with the equations for the velocities $\dot{x}, \ldots$ derived from (51), and are in complete accordance with the ones in [29]. Of course, the Cartesian components of the force need to be expressed in terms of the new variables (using the change of variables (51)).

As an illustrative example of the alternative procedure pointed out in section 6, let us solve the Gibbs–Appell equations (43) for the virtual displacement $\xi$. Using (52), the Gauss constraint restricted to $\text{Ker}_C[d_Tf]$ is

$$G^0(x,y,z,\nu^1,\nu^2,\omega^1,\omega^2) = \frac{1}{2}\left((\omega^1)^2 + (\omega^2)^2\right);$$

from the Gibbs–Appell equations $[\partial G^0/\partial \omega^d](\xi)\vartheta^d = \Phi_d\vartheta^d$ we obtain the components of the virtual displacement $\xi = \xi^d\partial/\partial \nu^d$:

$$\xi^1 = F_y, \quad \xi^2 = F_x + yF_z.$$

On the other hand, $b_0(x,v) = -d_Tf(\gamma_0(x,v)) = 2v_xv_y$ and therefore the fixed part of the dynamics $\tilde{\gamma}_0 = \gamma_0 + [d_Tf]^t \circ b_0$, restricted to $C$, is the admissible motion $\tilde{\gamma}_0(x,y,z,\nu^1,\nu^2) = (x,y,\nu^1,\nu^2,\alpha^1 = 0, \alpha^2 = y\nu^1\nu^2/(1 + y^2))$. Clearly, the constrained dynamics (24) coincides with (54).

Finally, instead of writing explicitly the Gibbs–Appell equations we may use the fact that $F\bar{G}^0 = H\bar{L}|_{\nu x} = \omega^1\vartheta^1 + (\omega^2/(1 + y^2))\vartheta^2$, from where and (53), equation (43) immediately yields (54).
6. Consider the system in $M = \mathbb{R}^3$ given by the Lagrangian $L = L_0 - mgz = m(v_x^2 + v_y^2 + v_z^2)/2 - mgz$ in terms of the usual Cartesian coordinates $(x, y, z)$ and velocities $(v_x, v_y, v_z)$, but subjected to the non-linear constraint

$$f = c^2(v_x^2 + v_y^2) - v_z^2.$$ 

Here $m$, $g$, and $c$ are positive constants.

This is a particular case of an example proposed by Appell [1], known as the ‘Appell particle’ and studied by several authors, see for instance [2, 28]; it is, however, a controversial example because of its non-physical realizability [33]. There are other examples of the same kind [3], and we want to show that the geometric method we have proposed works equally with non-linear constraints. Non-linear constraints can be originated also by an equivalent non-linear formulation of a linear problem (e.g. [2]) or by the action of some control devices [31, 39], or even by considering a constant of the motion as a constraint (e.g. the isokinetic dynamics).

Using coordinates $(x, y, z, v_x, v_y, v_z, a_x, a_y, a_z)$ in $T^2\mathbb{R}^3$, the inertial dynamics is null, i.e. $\gamma_0(x, \ldots, v_z) = (x, \ldots, v_z, a_x = 0, a_y = 0, a_z = 0)$, and obviously the force is $F = -mg\dot{v}_z$.

The constraint submanifold $C = f^{-1}(0)$ is described by the independent coordinates $(x, y, z, \nu_1, \nu_2)$, with the quasi-velocities $\nu_1 = v_x$ and $\nu_2 = v_y$. In other words, if we assume $v_z > 0$, then

$$C = \{(x, y, z, \nu_1, \nu_2)\} = \{(x, y, z, v_x = \nu_1, v_y = \nu_2, v_z = c\sqrt{\nu_1^2 + \nu_2^2}) \in T\mathbb{R}^3\}.$$ 

The coordinates in $TC$ are now $(x, y, z, \nu_1, \nu_2; u_x = \dot{x}, u_y = \dot{y}, u_z = \dot{z}, \alpha_1 = \dot{\nu}_1, \alpha_2 = \dot{\nu}_2)$, with the change for $v$-velocities and $a$-accelerations

$$v_x = \nu_1, \quad v_y = \nu_2, \quad v_z = c\sqrt{\nu_1^2 + \nu_2^2},$$

$$a_x = \alpha_1, \quad a_y = \alpha_2, \quad a_z = c\frac{\nu_1\alpha_1 + \nu_2\alpha_2}{\sqrt{\nu_1^2 + \nu_2^2}}.$$ 

Putting here $u = v$ we obtain $D_C$, with independent coordinates $(x, y, z, \nu_1, \nu_2, \alpha_1, \alpha_2)$, and the same transformation equations (55) for velocities and accelerations, because they do not depend on $u$. And putting $u = 0$, we obtain for $V\pi_C$ the coordinates $(x, y, z, \nu_1, \nu_2, \omega_1, \omega_2)$, with $\omega = \alpha$. The relations among the $w$-components of accelerations (in $V\tau_{\mathbb{R}^3}$) and the independent accelerations $\omega$ are the same ones:

$$w_x = \omega_1, \quad w_y = \omega_2, \quad w_z = c\frac{\nu_1\omega_1 + \nu_2\omega_2}{\sqrt{\nu_1^2 + \nu_2^2}}.$$ 

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As the applied force $F$ also obtain from here the explicit form of $\Gamma$ from where we see that the components of $(Gauss\ constraint)\ \nu \in \nu(\nu)$, its generalised inverse $[d_T f]^\dagger$ is the linear operator

$$[d_T f]^\dagger = \frac{1}{2c^4(v_x^2 + v_y^2)} \left( c^2 v_x \frac{\partial}{\partial v_x} + c^2 v_y \frac{\partial}{\partial v_y} - v_z \frac{\partial}{\partial v_z} \right).$$

As the applied force $F$ is $F = -mg \partial_z = H_L(-g \partial_v)$, its non-ideal part $F_{\text{nid}} = F - H_L \circ [d_T f]^\dagger \circ [d_T f] \circ H_L^{-1} F$ (32) turns out to be given by

$$F_{\text{nid}} = -\frac{mgc^2}{c^4(v_x^2 + v_y^2)}(v_x v_x \partial_x + v_y v_y \partial_y) + c^2(v_x^2 + v_y^2) \partial_z).$$

But on the submanifold $C$, according to (56), $\partial_x = \partial_{\nu^1}$, $\partial_y = \partial_{\nu^2}$ and $\partial_z = c(\nu^1 \partial_{\nu^1} + \nu^2 \partial_{\nu^2})/\sqrt{\nu^1)^2 + (\nu^2)^2}$, and finally the effective force is

$$F_{\text{nid}}|_C = -\frac{mgc}{\sqrt{(\nu^1)^2 + (\nu^2)^2}}(\nu^1 \partial_{\nu^1} + \nu^2 \partial_{\nu^2}). \quad (57)$$

Taking into account that the inertial motion is null $(a_0^x = a_0^y = a_0^z = 0)$, the Gauss constraint $G = m(w_x^2 + w_y^2 + w_z^2)/2$ gives the (extended) Gibbs function $G = m(a_x^2 + a_y^2 + a_z^2)/2$ (34), which transforms under (55) into:

$$G^0(x, y, z, \nu^1, \nu^2, \alpha^1, \alpha^2) = \frac{1}{2} m \left( (\alpha^1)^2 + (\alpha^2)^2 + c^2(\nu^1 \alpha^1 + \nu^2 \alpha^2)^2 \right). \quad (58)$$

According to (57) and (58), the Gibbs–Appell equations (49) for $\gamma(x, y, z, \nu^1, \nu^2) = (x, y, z, \nu^1, \nu^2, \Gamma^1, \Gamma^2)$ produce the following equations:

$$\frac{(1 + c^2)(\nu^1)^2 + (\nu^2)^2}{(\nu^1)^2 + (\nu^2)^2} \Gamma^1 + \frac{c^2 \nu^1 \nu^2}{(\nu^1)^2 + (\nu^2)^2} \Gamma^2 = -\frac{cg \nu^1}{\sqrt{(\nu^1)^2 + (\nu^2)^2}},$$
$$\frac{(\nu^1)^2 + (1 + c^2)(\nu^2)^2}{(\nu^2)^2 + (\nu^2)^2} \Gamma^2 + \frac{c^2 \nu^1 \nu^2}{(\nu^1)^2 + (\nu^2)^2} \Gamma^1 = -\frac{cg \nu^2}{\sqrt{(\nu^1)^2 + (\nu^2)^2}},$$

from where we see that the components of $\gamma$ satisfy the relation $\nu^1 \Gamma^2 - \nu^2 \Gamma^1 = 0$. We also obtain from here the explicit form of $\Gamma^1$ and $\Gamma^2$

$$\Gamma^1(x, y, z, \nu^1, \nu^2) = -\frac{gc}{1 + c^2} \frac{\nu^1}{\sqrt{(\nu^1)^2 + (\nu^2)^2}},$$
$$\Gamma^2(x, y, z, \nu^1, \nu^2) = -\frac{gc}{1 + c^2} \frac{\nu^2}{\sqrt{(\nu^1)^2 + (\nu^2)^2}}.$$
Finally, the equations of motion in the coordinates \((x, y, z, \nu^1, \nu^2)\) are:

\[
\begin{align*}
\dot{\nu}^1 &= -\frac{g c}{1 + c^2} \frac{\nu^1}{\sqrt{(\nu^1)^2 + (\nu^2)^2}}, \\
\dot{\nu}^2 &= -\frac{g c}{1 + c^2} \frac{\nu^2}{\sqrt{(\nu^1)^2 + (\nu^2)^2}},
\end{align*}
\]

(59)

together with \(\dot{x} = \nu^1, \dot{y} = \nu^2, \dot{z} = c\sqrt{(\nu^1)^2 + (\nu^2)^2}\). These equations are the same as the ones obtained by means of another different methods, see e.g. [28]. On the other hand, the equations (59) are very familiar in elementary mechanics and are analogous to those of the motion of a unit mass moving on a horizontal surface with a Coulomb friction of coefficient \(\mu = c/(1 + c^2)\). This relation allows us to interpret the complicated motion of Appell’s particle. We also see clearly in (59) that \(\dot{\nu}^1/\nu^1 = \dot{\nu}^2/\nu^2\), as indicated above.

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**References**


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