GEOMETRIC STUDY OF HAMILTON’S VARIATIONAL PRINCIPLE

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Abstract. Higher order tangent bundle geometry and sections along maps are used in Geometrical Mechanics in order to develop an intrinsic variational calculus. The role of variational derivative as the bundle operator associated to exterior differential on the set of trajectories is remarked. Euler–Lagrange equations and Poincaré–Cartan form are redervied in this way. Helmholtz conditions for the inverse problem of Lagrangian Mechanics are geometrically obtained for the general higher order case.

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1. Introduction.

The geometric formulation of Classical Mechanics, in both Lagrangian and Hamiltonian approaches, has been a subject of most interest in past years as an important front in the process of geometrization of many branches of Physics, and in particular, as a first step in the theory of Geometric Quantization. Classical Mechanics can now be considered as a chapter of the theory of Symplectic Geometry, or Presymplectic Geometry when singular Lagrangians are considered [1]. A deep study of the geometry of tangent and cotangent bundles (velocity phase spaces and phase spaces, respectively) has been necessary for this development of the Lagrangian approach and some geometric objects as the total time derivative, vertical endomorphism, Liouville vector field, SODE (second-order differential equations) vector fields, Poincaré–Cartan one-form, etc., appear as the basic ingredients of the theory, together with the dynamical equation that in the Lagrangian case turns out to be

\[ i(\Gamma_L)\omega_L = dE_L \]  

which is a geometric version of the Euler–Lagrange equations when \( L \) is regular.

This dynamical equation has a Hamiltonian counterpart in the cotangent bundle of the configuration space \( M \). The Legendre map \( FL : TM \to T^*M \) is a diffeomorphism when \( L \) is hyper-regular and the symplectic form \( \omega_L \) is the \( FL \)-pull-back of the canonical symplectic structure \( \omega_c \) on the phase space \( T^*M \). Moreover, a Hamiltonian function \( H \in C^\infty(T^*M) \) associated to the Energy function \( E_L \) is defined by \( H \circ FL = E_L \) and a key stone of the geometric theory is the parallelism between both geometric equations, (1.1) on \( TM \) and

\[ i(Y_H)\omega_c = dH \]  

on \( T^*M \) respectively [1].

The vector fields solution of each dynamical equation are \( FL \)-related, i.e., \( Y_H = FL_*(\Gamma_L) \), and therefore the spaces of solution trajectories are equivalent. The Euler–Lagrange equations are presented by means of the solution vector field \( \Gamma_L \), as the differential equations are satisfied by the curves in configuration space projection from integral curves of this vector field.

However, we feel that some important points have not been clarified yet, the first one being that vector fields are not the unique way to represent geometrically differential equations, and in fact they are not the most natural way, for example when considering differential equations that cannot be put in normal form. On the contrary, a more general and perhaps natural geometric representation of differential equations is given as being submanifolds of higher-order tangent bundles [6] (SODE’s are submanifolds of \( T^2M \), for example). In this line, the Euler–Lagrange equations for first-order Lagrangians may be first considered as a submanifold of \( T^2M \), and only as a second step the associated vector field on \( TM \) is obtained through the natural injection \( i_{1,1} : T^2M \to T(TM) \).
The second point to be remarked here is that, although many works about variational calculus can be found in the literature \[7, 10, 12, 13, 14\], a really direct and explicit derivation of Euler–Lagrange equations from Hamilton’s variational principle has not been made geometrically, at least for the general case of higher-order Lagrangians.

We shall take advantage of a great development during the last few years of the geometry of higher-order tangent bundles \[3\] and the introduction in Mechanics of sections along maps as a basic and powerful concept \[4, 5\]; these are the tools we will use in this paper in order to present an explicit geometric formulation of variational calculus and a rederivation of Euler–Lagrange equations in different presentations. The use of higher-order tangent bundle geometry allows us to develop the general case of higher-order Lagrangian functions while the concept of sections along a map appears as a very natural tool in the geometric formulation of variational calculus, as it happens to be for other problems of Classical Mechanics, for example Noether’s theorem \[4\].

The output of this work is the representation of Euler–Lagrange equations through a one-form along the projection. More specifically, for first-order Lagrangians, \(\alpha_L : T^2M \rightarrow T^*M\) is one-form along \(\tau_2\) in such a way that the set of points on \(T^2M\) where this one-form vanishes is the submanifold representing the Euler–Lagrangian equations. The geometric expression of this one-form by means of the variational derivative operator \[13, 14\] can be immediately related to the usual representation of the Euler–Lagrange equations through the symplectic dynamical equation on Hamiltonian systems.

An application showing the interest of this approach is, for example, to clarify the meaning of the variational derivative operator as the geometric counterpart of the exterior differential on the infinite dimensional manifold of curves on the configuration space \[8\] (so that it can be applied to general forms of arbitrary degree and not only one-forms). This fact allows us to formulate the inverse problem of Lagrangian Mechanics in a very compact geometric way. Another interesting point would be the generalization of this approach to the case of Field Theories, in the framework of Jet bundles, because the Euler–Lagrange equations are geometrically well defined for the general higher-order case, even if there is non-uniqueness on the Poincaré–Cartan form \[2\]. The explicit construction of the variational derivative in this case may help us to clarify some aspects of the formulation in higher-order field theories, in both Lagrangian and Hamiltonian approaches.

2. Notation and basic definitions.

The notation to be used in this paper follows mainly that of \[3\]. Given a differentiable manifold \(M\), the \(k\)-th-order tangent bundle is denoted \(T^kM\) (the first-order tangent bundle \(T^1M\) is usually denoted \(TM\)) and represents the set of equivalence classes of curves on \(M\) with a \(k\)-th degree contact. Natural projections \(\tau^l_k : T^kM \rightarrow T^lM\), for \(l \leq k\), as well as injections \(i_{k,l} : T^{k+l}M \rightarrow T^k(T^lM)\) may be defined. In order to simplify the expressions we usually will not use a different notation for objects living in a lower tangent bundle and that can be pulled-back to a higher one, as, for example, a one-form.
\( \alpha \in \bigwedge (T^lM) \) and its pull back to \( T^kM \), \( \tau^{i_*}_k(\alpha) \). The case \( i_{1,l} : T^{l+1}M \to T(T^lM) \) plays an important role in the identification of a \((l+1)\)-th differential equation with its representing vector field on \( T^lM \). This injection \( i_{1,l} \) is also denoted \( d_T(\cdot) \) or sometimes simply \( d_T \), without subindex \( l \), showing by the context which particular injection we are dealing with, and called total time derivative. It can also be seen as a vector field along the projection \( \tau^{i}_{l+1} \). We recall that a vector field \( X \) along a map \( \phi : M \to N \) is a map \( X : N \to TM \) over \( \phi \), i.e., \( \tau_M \circ X = \phi \). The set of vector fields along \( \phi \) is denoted \( X(\phi) \) and, as the usual vector fields on manifolds, vector fields along maps define first-order differential operators acting on functions by \( X : C^\infty(M) \to C^\infty(N) \) such that \( X(f)(n) = \langle X(n) | df(\phi(n)) \rangle \) for every \( f \in C^\infty(M) \) and \( n \in N \); for example, for every function \( f \in C^\infty(T^lM) \), \( d_Tf \) is a function in \( C^\infty(T^{l+1}M) \) given by

\[
\forall u \in T^{l+1}M, \quad d_T f(u) = \langle df | i_{1,l}(u) \rangle
\]

A vector field along the map \( \phi \), \( X \in \mathfrak{X}(\phi) \), defines a map \( i_X : \bigwedge (M) \to \bigwedge (N) \) given by

\[
(i_X \alpha)(n) = \phi^*(n)[i(X(n))\alpha(\phi(n))].
\]

Given a local coordinate system \((U, x^i)\) on \( M \), there is an induced coordinate system \((V, x^i_l)\) on \( T^kM \), with \( V = \tau^{-1}_k(U) \) and \( x^i_l = d_T x^i \) for \( l \leq k \). In this coordinate system the injection \( i_{1,k} \) is given by

\[
i_{1,k}(x^i, x^i_1, \ldots, x^i_{k+1}) = (x^i, x^i_1, \ldots, x^i_k; x^i_1, x^i_2, \ldots, x^i_{k+1})
\]

and the vector field along \( \tau^k_{k+1} \), \( d_T \) is represented by

\[
d_T = \sum_{l=0}^{k} x^i_{l+1} \frac{\partial}{\partial x^i_l}
\]

with \( x^i_0 = x^i \).

Other basic geometric objects are the vertical endomorphism \( S \) in \( T^{k+1}M \), a \((1,1)\)-type tensor field, given in local coordinates by

\[
S = \sum_{l=0}^{k} (l + 1) \frac{\partial}{\partial x^i_{l+1}} \otimes dx^i_l,
\]

and the Liouville or dilation vector field \( \Delta \)

\[
\Delta = \sum_{l=1}^{k+1} lx^i_l \frac{\partial}{\partial x^i_l}.
\]
Once again, for the sake of simplifying notation, we do not use a subindex to indicate in which particular tangent bundle $S$ or $\Delta$ are defined, but the context will show the appropriate index.

The coordinate representation of a system of $k$-th-order differential equations

$$f^a(x, x_1, \ldots, x_k) = 0$$

is in fact but a local definition of a submanifold $D_k$ of $T^kM$, and when the system can be written in normal form $x^i_k = \Phi^i(x, x_1, \ldots, x_{k-1})$, the corresponding submanifold $D_k$ is the image of a section $\sigma$ for $\tau_{k-1}^k$, i.e., $\sigma : T^{k-1}M \to T^kM$, is such that $\tau_{k-1}^k \circ \sigma = \text{id}_{T^{k-1}M}$. In this case, the composition of $\sigma$ with the natural injection $i_{1,k-1}$ defines a vector field $\Gamma$ on $T^{k-1}M$, with coordinate expression

$$\Gamma = \sum_{l=0}^{k-2} x^i_{l+1} \partial x^i_l + \Phi^i \frac{\partial}{\partial x^i_{k-1}},$$

which is called a $k$-ODE ($k$-th-order differential equation) vector field.

Given a system of $k$-th-order differential equations $D_k \subset T^kM$, the curves of $M$ solving the system are those curves $\gamma$ in $M$ that when lifted to $T^kM$ take values in $D_k$. In local coordinates, for $\gamma : \mathbb{R} \to M$ given by $x^i(t)$, its lift $\gamma_k : \mathbb{R} \to T^kM$ is given by $(\gamma_k(t))^i_l = \frac{d^i}{dt^l} x^i(t)$ and when it takes values in $D_k$,

$$f^a(x(t), \frac{d}{dt} x, \ldots, \frac{d^k}{dt^k} x) = 0$$

and the local equations are satisfied for $\gamma$. In the case of differential equations written in normal form it is equivalent to say that the $(k-1)$-th lifting of the curve is an integral curve of the representing vector field $\Gamma$,

$$\dot{\gamma}_{k-1}(t) = \Gamma(\gamma_{k-1}(t))$$

This kind of $k$-ODE vector fields $\Gamma$, representing $k$-th-order differential equations, also called semisprays and characterized by $S(\Gamma) = \Delta$, are such that their integral curves on $T^{k-1}M$ are always a lifting of a curve in the base $M$, solutions of the $k$-th-order differential equation represented by $\Gamma$.

We recall that given a first-order Lagrangian, a function $L \in C^\infty(TM)$, the Poincaré–Cartan form is defined by $\theta_L = S^* (dL)$ ($S^*$ is used when the tensor field $S$ is applied on forms instead of on vector fields), and the Cartan two-form $\omega_L = - d\theta_L$ is symplectic for regular Lagrangians. The energy function is defined by $E_L = \Delta(L) - L \in C^\infty(TM)$. All these objects allow us to write the well known dynamical equation $[1]$

$$i(\Gamma_L)\omega_L = dE_L$$
For regular Lagrangians it happens that the unique vector field $\Gamma_L$ solution of this geometric equation represents the Euler–Lagrange differential equations associated to $L$. Therefore, this dynamical equation has been used in the literature as the geometric representation of Euler–Lagrange equations for $L$, although sometimes it is presented in an alternative way, only equivalent for regular Lagrangians, $\mathcal{L}_{\Gamma}\theta_L = dL$, where $\mathcal{L}_{\Gamma}$ denotes the Lie derivative along the vector field $\Gamma_L$. An important remark is the fact that for singular Lagrangians, when the two-form $\omega_L$ is not symplectic, there is not equivalence anymore between the Euler–Lagrange equations and the former dynamical equation.

As we pointed out in the introduction, the natural representation of Euler–Lagrange equations is a submanifold of $T^2M$ which is obtained using the variational derivative, a differential operator acting on one-forms defined by [13, 14]

$$\delta = \text{id} - d_T \circ S^*$$

with id denoting the identity operator. When $\delta$ is applied to the differential of the Lagrangian we obtain a one-form $\alpha_L = \delta(dL)$ which is defined on $T^2M$ (remember that $d_T$ is a vector field over the projection $\tau_2^1$) and it happens to be semibasic $S^*(\alpha_L) = 0$, so that $\alpha_L$ can be equivalently seen as a one-form along the projection $\tau_2$, $\tilde{\alpha}_L : T^2M \to T^*M$.

The inverse image of the zero section for $\pi : T^*M \to M$ along $\tilde{\alpha}_L$ defines a submanifold $D_L$ of $T^2M$ representing the Euler–Lagrange equations for $L$

$$D_L = \{u \in T^2M \mid \tilde{\alpha}_L(u) = 0\}$$

for both regular and singular Lagrangians. For regular Lagrangians, $D_L$ is the image of a section $\sigma_L$ along $\tau_2^1$, and the $\sigma_L$-pull-back of $\tilde{\alpha}_L$ onto $TM$ reproduces the dynamical equation

$$\sigma_L^*(\tilde{\alpha}_L) = 0 = dL - \mathcal{L}_{\Gamma_L}\theta_L$$

with $\Gamma_L = i_{1,1} \circ \sigma_L \in \mathfrak{X}(TM)$ being a vector field on $TM$.

The aim of this paper is to obtain an explicit geometric derivation of the variational derivative operator $\delta$ for the general higher-order case, starting from Hamilton’s Variational Principle, formulated in geometric language, and developing the geometric variational calculus. The next lines are devoted to introduce some preliminary definitions and notation on this subject.

Let us first consider the set of curves on the manifold $M$

$$\mathcal{C} = \{\gamma : \mathbb{R} \to M \mid \gamma \text{ differentiable}\}$$

which can be seen as an infinite-dimensional manifold. In the formulation of Hamilton’s Variational Principle some subsets of fixed end-points curves are used, $\mathcal{C}_I = \{\gamma : I \subset \mathbb{R} \to M \mid \gamma(t_a) = x_a, a = 1, 2\}$, where $I = [t_1, t_2] \subset \mathbb{R}$ is a closed interval, and $x_1, x_2$ are
fixed points on $M$. Given a curve $\gamma \in \mathfrak{C}_I$, first-order variations of this curve are vectors $X \in T_\gamma \mathfrak{C}_I$, but there is an alternative description of $X$ in terms of a vector field along the map $\gamma$, also denoted by $X$,

$$X : I \to TM \text{ such that } \tau \circ X = \gamma \text{ and } X(t_a) = 0$$

which allows us to avoid the technical problems of infinite-dimensional manifolds and its tangent spaces. The idea is that when we consider a one parameter variation $\gamma_\epsilon$ of the curve $\gamma$, $\gamma_{\epsilon=0} = \gamma$, with the fixed end-points condition, $\gamma_\epsilon(t_a) = x_a$, i.e., a “curve” on $\mathfrak{C}_I$ beginning at $\gamma$, we obtain a one parameter set of curves $\rho_\epsilon(\epsilon) = \gamma_\epsilon(t)$ on $M$, and the tangent vectors to these curves, $\frac{d}{d\epsilon} \rho_t$, for every fixed $t$ on $I$, represent the first-order variation of $\gamma$. But this set of vectors $\rho_t(0)$ is just a vector field along the curve $\gamma$, $X : I \to TM$, given by $X(t) = \rho_t(0)$.

In local coordinates, given $\gamma = \{x^i(t)\}$ and a one parameter variation $\gamma_\epsilon = \{x^i(t) + \epsilon \psi^i(t)\}$, the vector field along $\gamma$ representing this first-order variation is

$$X(t) = \psi^i(t) \frac{\partial}{\partial x^i |_{\gamma(t)}} \in T_{\gamma(t)}M$$

and fixed end-point conditions give $X(t_a) = 0$.

A basic ingredient in the local calculus of variations is the fact that for the variation in coordinates $\delta x^i(t) = \epsilon \psi^i(t)$, the induced variation of the velocities $\delta \frac{dx^i}{dt}$ is obtained by commuting the variation operator $\delta$ with the time derivative operator $\frac{d}{dt}$, so that

$$\delta v^i(t) = \epsilon \frac{d}{dt} \psi^i(t),$$

and integration by parts can be applied in the variation of the action integral.

The geometric counterpart of this construction is the lifting of vector fields along curves [4, 5]. Given a vector field along the curve $\gamma$, $X$, the lifted vector field along $\gamma_1$, $X_1$, is the map $X_1 : I \to T(TM)$ such that

(i) $\tau_{TM} \circ X_1 = \gamma_1 : I \to TM$

i.e., the lifted vector field $X_1$ is a vector field along the lifted curve $\gamma_1$

(ii) $\tau_* \circ X_1 = X$

and

(iii) $\frac{d}{dt} \circ X(f) = X_1 \circ d_T(f)$

for every $f \in C^\infty(M)$, where $X$ and $X_1$ are to be seen as acting on functions in the last formula.
In local coordinates, given \( X(t) = \psi^i(t) \partial_{x^i} |_{\gamma(t)} \), properties (i) and (ii) above imply
\[
X_1(t) = \left( \psi^i(t) \frac{\partial}{\partial x^i} + \chi^i(t) \frac{\partial}{\partial x^1_i} \right) \bigg|_{\gamma_1(t)},
\]
and taking \( f = x^i \) a coordinate function, property (iii) gives
\[
\frac{d}{dt} \psi^i(t) = X_1(d_T x^i) = X_1(x^1_i) = \chi^i(t)
\]
so that finally
\[
X_1 = \left( \psi^i(t) \frac{\partial}{\partial x^i} + \frac{d}{dt} \psi^i(t) \frac{\partial}{\partial x^1} \right) \bigg|_{\gamma_1(t)}.
\]

A repeated application of this property allows us to define a higher-order lift \( X_k : I \to T(T^k M) \), i.e., a vector field along the lifted curve \( \gamma_k \), satisfying
\[
\frac{d}{dt} \circ X_{k-1}(f) = X_k \circ d_T(f) \quad \forall f \in C^\infty(T^{k-1} M).
\]
Its coordinate expression is given by
\[
X_k(t) = \left( \psi^i(t) \frac{\partial}{\partial x^i} + \sum_{l=1}^k \frac{d^l}{dt^l} \psi^i(t) \frac{\partial}{\partial x^1_i} \right) \bigg|_{\gamma_k(t)}.
\]

Lifts of vector fields along curves are closely related to the more usual concept of complete lift of a true vector field on a manifold. In fact, given an arc \( I \) of a curve \( \gamma \) on \( M \) with no selfintersection and a vector field \( X \) along the curve, \( X \in \mathfrak{X}(\gamma) \), we can choose a vector field \( Y \) on \( M \) whose restriction to the curve is but \( X, Y \circ \gamma |_I = X |_I \). The lifting \( X_1 \) of \( X \) and the complete lift \( Y^c \) of \( Y \) to \( TM \) are similarly related, i.e., \( Y^c \circ \gamma_1 |_I = X_1 |_I \), and this is also true for higher-order liftings. This fact is easily checked on local coordinates taking into account that the restriction of the total time derivative \( d_T \) to a curve \( \gamma \) is just the derivative with respect to the parameter of the curve:
\[
\forall f \in C^\infty(M) \quad d_T(f)(\gamma(t)) = \frac{d}{dt} \gamma^*(f)
\]
Complete lifts \( Y^c \) of vector fields \( Y \in \mathfrak{X}(M) \), usually defined by the natural lift of the associated flow, can also be characterized by properties:
\[
\tau_* (Y^c) = Y \quad \text{(to be compared with (ii))}
\]
and
\[
i_{Y^c} \circ d_T |_{\Lambda(M)} = d_T \circ i_Y |_{\Lambda(M)} \quad \text{(to be compared with (iii))},
\]
where \( d_T \) is seen here as a derivation on forms, commuting with the exterior differential \( d_T \circ d = d \circ d_T \) [3]. Similarly, higher-order complete lifts \( Y_k \in \mathfrak{g}(T^k M) \) are characterized by iterating the former properties

\[
\tau_k^{k-1} \ast (Y_k) = Y_{k-1} \quad \text{and} \quad i_{Y_k} \circ d_T = d_T \circ i_{Y_{k-1}}.
\]

The coordinate representation of \( Y_k \), for a vector field \( Y = \chi^i(x) \frac{\partial}{\partial x_i} \) is

\[
Y_k = \sum_{l=0}^{k} [d_T^l \chi^i(x)] \frac{\partial}{\partial x^i_l}
\]

with \( d_T^0 \chi^i(x) = \chi^i(x) \).

Another property playing a crucial role in the geometric calculus of variations is the explicit expression of the complete lift of the vector fields \( fY \in \mathfrak{g}(M) \), with \( f \) an arbitrary function on \( M \). More specifically, if \( f \in C^\infty(M) \) and \( Y \in \mathfrak{g}(M) \), then

\[
(fY)_k = \sum_{l=0}^{k} \frac{1}{l!} \tau^*_k((d_T^l f)S_l(Y_k)) = \tau^*_k(fY_k) + \tau^*_{k-1}(d_T^l f)S_{k-1}(Y_k) + \cdots
\]

\[
+ \frac{1}{(k-1)!} \tau^*_{k-1}(d_T^{k-1} f)S_{k-1}(Y_k) + \frac{1}{k!} (d_T^k f)S_k(Y_k)
\]

which can be obtained from the defining properties of complete lifts (see [5] for an intrinsic proof). A corresponding property for lifts of vector fields along curves does exist. So, if \( f \in C^\infty(I) \) and \( X \in \mathfrak{g}(\gamma) \), then \((fX)_k\) is the vector field along the lifted curve \( \gamma_k \)

\[
(fX)_k = \sum_{l=0}^{k} \frac{1}{l!} \frac{d^l f}{dt^l} S_l(X_k).
\]


Let us consider a Lagrangian function \( L \in C^\infty(TM) \) on the velocity phase space. The action integral \( \mathcal{A} = \int \gamma_1^* (L) dt \) is a functional on the set of the trajectories on \( M \), i.e., a function on the manifold \( \mathcal{C} \) of curves on \( M \). When choosing a subset of fixed end-points curves,

\[
\mathcal{C}_I = \{ \gamma : I \subset \mathbb{R} \to M \mid \gamma(t_a) = x_a, \quad a = 1, 2 \}
\]

Hamilton’s variational principle picks out the curves making the action integral stationary, or in other words, it looks for points of \( \mathcal{C}_I \) where \( d\mathcal{A} = 0 \). Here \( \mathcal{A} : \mathcal{C}_I \to \mathbb{R} \) is given by

\[
\mathcal{A}(\gamma) = \int_I \gamma_1^* (L) dt = \int_{t_1}^{t_2} L(x(t), v(t))dt.
\]
Let us suppose that \( \gamma_0 \in \mathcal{C}_I \) is one of these curves making \( \mathcal{A} \) stationary. This means that for an arbitrary first-order variation of \( \gamma_0 \), \( \gamma_\epsilon(t) = x^i(\epsilon, t) = x^i_0(t) + \epsilon \eta^i(t) \), the first-order variation of \( \mathcal{A} \) vanishes

\[
\delta_\epsilon \mathcal{A} = \frac{d}{d\epsilon} \mathcal{A}(\gamma_\epsilon)|_{\epsilon=0} = 0.
\]

The variation is represented in terms of the vector field along \( \gamma_0 \), \( X \in \mathfrak{X}(\gamma_0) \equiv T_{\gamma_0} \mathcal{C}_I \) by

\[
X = \eta^i(t)(\partial/\partial x^i)|_{\gamma_0(t)},
\]

and therefore the condition for the action integral to be stationary is

\[
\langle d\mathcal{A}(\gamma_0) \mid X \rangle = 0 \quad (3.1)
\]

for arbitrary \( X \in T_{\gamma_0} \mathcal{C}_I \).

Let us consider a given variation vector field \( X \in \mathfrak{X}(\gamma_0) \) and a partition of the curve \( \gamma_0 \) in pieces \( I_\alpha, \bigcup I_\alpha = I \), such that the curve does not have any self-intersection in each piece. We can choose in each piece a vector field \( Y_\alpha \) on \( M \) whose restriction to \( \gamma_0(I_\alpha) \) coincides with the given \( X \). For this fixed piece \( I_\alpha \) let \( \phi_\epsilon^{(\alpha)} \) be the integral flow of \( Y_\alpha \).

The variation curve \( \gamma_\epsilon \) given by \( \gamma_\epsilon|_{I_\alpha} = \phi_\epsilon^{(\alpha)} \circ \gamma_0 \mid_{I_\alpha}, \gamma_\epsilon(t) = \phi_\epsilon^{(\alpha)}[\gamma_0(t)] \), is represented by \( X \) in first-order because \( X \) and \( Y_\alpha \) coincide on the curve. Equation (3.1) can now be rewritten

\[
\delta_\epsilon \mathcal{A} = 0 = \sum_\alpha \frac{d}{d\epsilon} \int_{I_\alpha} (\phi_\epsilon^{(\alpha)^c} \circ \gamma_{01})^*(L) \, dt
\]

because the lift \( \gamma_\epsilon^{1c} \) can be decomposed as \( \gamma_\epsilon^{1c} \mid_{I_\alpha} = \phi_\epsilon^{(\alpha)^c} \circ \gamma_{01} \mid_{I_\alpha} \).

Then, using the definition of Lie derivative we finally obtain

\[
\frac{d}{d\epsilon} \mathcal{A}(\gamma_\epsilon) = 0 = \sum_\alpha \int_{I_\alpha} \gamma_{01}^*(\mathcal{L}_{Y_\alpha} L) \, dt = \sum_\alpha \int_{I_\alpha} \gamma_{01}^*(Y_\alpha^c(L)) \, dt = \int_I X_1(L) \, dt.
\]

Obviously, the final result does not depend on the choices of \( Y_\alpha \) out from the curve \( \gamma_0 \).

In local coordinates, the variation of the action integral takes the form

\[
\delta_\epsilon \mathcal{A} = \int_{t_1}^{t_2} \left\{ \eta^i(t) \frac{\partial L}{\partial x^i}(x(t), v(t)) + \left[ \frac{d}{dt} \eta^i(t) \right] \frac{\partial L}{\partial v^i}(x(t), v(t)) \right\} \, dt
\]

and by means of an integration by parts on the second term, and taking into account the arbitrariness of the functions \( \eta^i(t) \) and its vanishing on the end-points, \( \eta^i(t_a) = 0 \), we will get the Euler-Lagrange equations

\[
\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial v^i} = 0.
\]
A parallel calculus can be made in an intrinsic way by taking a first-order variation $\bar{f}X \in \mathfrak{X}(\gamma_0)$, with $f \in C^\infty(M)$ being an arbitrary function and $\bar{f} = \gamma_0^*(f)$, i.e. $(\bar{f}X)(t) = f(\gamma_0(t))X(t)$. The variation of the action integral becomes

$$
\langle dA(\gamma_0) \mid \bar{f}X \rangle = \int_{t_1}^{t_2} (\bar{f}X)_1(L) \, dt = \int_{t_1}^{t_2} \left\{ \bar{f}X_1(L) + \left( \frac{d}{dt} \bar{f} \right) S(X_1)(L) \right\} \, dt.
$$

An integration by parts leads to

$$
0 = \int_{t_1}^{t_2} \bar{f} \left\{ X_1(L) - \frac{d}{dt}[S(X_1)(L)] \right\} \, dt + [\bar{f}(t_2)S(X_1)(L)(t_2) - \bar{f}(t_1)S(X_1)(L)(t_1)],
$$

the last two terms being identically zero by the fixed end-points condition $X(t_a) = 0$, which implies $S(X_1)(t_a) = 0$. The arbitrariness of the function $f$ allows us to rewrite the above condition as

$$
X_1(L) - \frac{d}{dt} \circ i_{X_1} \circ S^*(dL) = 0.
$$

If we make use of the relation $\frac{d}{dt} \circ i_{X_1} = i_{X_2} \circ d_T$, it can also be expressed as

$$
i_{X_2}[\text{id} - d_T \circ S^*](dL) = 0
$$

which is an identity along the image of the second lift of the curve $\gamma_0$.

The one form $\alpha_L = [\text{id} - d_T \circ S^*](dL) \in \bigwedge(T^2M)$ is semibasic $S^*(\alpha_L) = 0$, as can be easily checked using the property $[S^*, d_T]|_{\bigwedge^1} = \text{id}|_{\bigwedge^1}$ [3], and the arbitrariness of the variation vector field $X$ gives us the geometric representation of Euler-Lagrange equations

$$
\delta(dL)_{\mid \gamma_{02}} = 0
$$

with $\delta = \text{id} - d_T \circ S^*$ the variational derivative operator.

Therefore, the one form

$$
\alpha_L = \left[ \frac{\partial L}{\partial x^i} - d_T \left( \frac{\partial L}{\partial v^i} \right) \right] dx^i
$$

vanishes on every point on the image of $\gamma_{02} : I \to T^2M$, i.e., vanishes on the subset $D_L \subset T^2M$ defining the Euler-Lagrange equations for $L$. The curves $\gamma_0$ making the action integral $\mathcal{A}$ stationary are those curves in $M$ whose second lifting take values on $D_L = \{ u \in T^2M \mid \alpha_L(u) = 0 \}$.  


4. The Higher-Order Case

Once the first-order case has been developed with some detail we can apply the same construction to higher-order Lagrangian Mechanics, which happens to be quite similar, but for the appearance of extra terms on the variational derivative and the modification of the initial data to be specified for the trajectory.

Let us consider a Lagrangian function depending on up to \( k \)-th degree derivatives \( L \in C^\infty(T^k M) \). Euler-Lagrange equations for this \( L \) are in general \( 2k \)-th-order differential equations, which means that we must fix \( 2k \) boundary conditions, the positions and up to \( (k - 1) \)-th degree derivatives on the endpoints of the trajectories. Therefore, on the set \( \mathcal{C} \) of curves on \( M \), we consider subsets of fixed endpoints

\[
\mathcal{C}_I = \{ \gamma : I = [t_1, t_2] \rightarrow M \mid \gamma_{k-1}(t_a) = u_a \quad u_a \in T^{k-1} M \} .
\]

The aim of Hamilton’s Variational Principle is to find the elements of \( \mathcal{C}_I \) making the action integral \( A : \mathcal{C}_I \rightarrow \mathbb{R} \), given by

\[
A(\gamma) = \int_I \gamma^* k(L) dt ,
\]

stationary, i.e., to find curves \( \gamma_0 \in \mathcal{C}_I \) such that \( \langle dA(\gamma_0) \mid X \rangle = 0 \) for an arbitrary first-order variation \( X \in T_{\gamma_0} \mathcal{C}_I \). Boundary conditions on the curves of \( \mathcal{C}_I \) impose conditions to the variation vectors \( X \in T_{\gamma} \mathcal{C}_I \); the corresponding vector fields along the curves \( X \in \mathfrak{x}(\gamma_0) \), \( X(t) = \psi^i(t)(\partial/\partial x^i)_{|\gamma_0(t)} \), are such that \( \psi^i(t_a) = 0 \) and \( \frac{d^l}{dt^l} \psi^i(t) \mid_{t=t_a} = 0 \) for every \( 1 \leq l \leq k - 1 \). These conditions can be rewritten in geometric terms as

\[
X_{k-1}(t_a) = 0
\]

or, equivalently,

\[
(\tau^{k-1}_l)_* X_i(t_a) = 0 \quad \forall l > k - 1
\]

which asserts that \( l \)-th-order lifts of \( X \) are \( (\tau^{k-1}_l) \)-vertical on the endpoints.

The first-order variation associated to \( X \) on the action integral takes the form

\[
\langle dA(\gamma_0) \mid X \rangle = \int_I X_k(L) dt
\]

For a new first-order variation \( \tilde{f}X \in \mathfrak{x}(\gamma_0) \), with \( f \in C^\infty(M) \) an arbitrary function and \( \tilde{f} = \gamma_0^*(f) \) (note that boundary conditions for \( \tilde{f}X \) are fulfilled too), the variation of the action integral is

\[
\langle dA(\gamma_0) \mid \tilde{f}X \rangle = \sum_{l=0}^{k} \frac{1}{l!} \int_I \frac{d^l}{dt^l} \{ S^l(X_k)(L) \} dt
\]
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An iterated integration by parts in order to isolate an integral term factorized by $\bar{f}$ leads to

$$\langle dC(\gamma_0) | \bar{f}X \rangle = \sum_{l=0}^{k} \frac{(-1)^l}{l!} \int_I \bar{f} \frac{d^l}{dt^l} \{ i(X_k) S^* l(dL) \} \, dt +$$

$$\left\{ \sum_{l=1}^{k} \frac{1}{l!} \sum_{s=0}^{l-1} (-1)^s \frac{d^{l-s-1}}{dt^{l-s-1}} \frac{d^s}{dt^s} \{ S^l (X_k) (L) \} \right\}_{t_2}^{t_1}$$

where $\big|_{t_1}^{t_2}$ on the second term means that it is evaluated on both endpoints and then subtracted.

Using repeatedly the relation $\frac{d}{dt} \circ i_{X_l} = i_{X_{l+1}} \circ d_T$, the boundary terms can be appropriately manipulated to become

$$\left\{ \sum_{l=1}^{k} \frac{1}{l!} \sum_{s=0}^{l-1} (-1)^s \frac{d^{l-s-1}}{dt^{l-s-1}} \frac{d^s}{dt^s} \{ i_{X_{k+s}} \circ d_T \circ S^* l(dL) \} \right\}_{t_2}^{t_1}$$

Now, taking into account that $s$ is always lower than $l$ it is not hard to see that the one-form $\beta = d_T^* [S^* l(dL)]$ has only terms up to $dx_{k-1}$, i.e., $S^* k (\beta) = 0$,

$$S^* l(dL) = a_0 \, dx + \cdots + a_{k-l} \, dx_{k-l}$$

$$d_T^* S^* l(dL) = b_0 \, dx + \cdots + b_{k-l+s} \, dx_{k-l+s}$$

and this, together with the boundary conditions on $X$ imply the vanishing of those boundary terms.

Using again the property $\frac{d}{dt} \circ i_{X_l} = i_{X_{l+1}} \circ d_T$ on the integrand, we will obtain

$$\langle dA(\gamma_0) | \bar{f}X \rangle = \sum_{l=0}^{k} \frac{(-1)^l}{l!} \int_I \bar{f} \{ i(X_{k+l}) \circ d_T \circ S^* l(dL) \} \, dt$$

and the arbitrariness of $\bar{f}$ shows that for every stationary point $\gamma_0$ for $A$ on $C_I$, the condition $dA(\gamma_0) = 0$ is equivalent to

$$\sum_{l=0}^{k} \frac{(-1)^l}{l!} i(X_{k+l}) \{ d_T^* \circ S^* l(dL) \} = 0$$

along the image of the $2k$-th lifting of $\gamma_0$, namely $\gamma_{02k}$. The one-form $\alpha_L = \sum_{l=0}^{k} \frac{(-1)^l}{l!} d_T^* \circ S^* l(dL) \in \wedge^1 (T^{2k}M)$ is, as in the first-order case, semibasic, $S^* (\alpha_L) = 0$, so that it can
alternatively be seen as a one-form along the projection map \( \tau_{2k} \), \( \tilde{\alpha}_L; T^{2k}M \to T^*M \); the former equation can be rewritten with the notation

\[
\delta(dL) |_{\gamma_{0,2k}} = 0
\]

where \( \delta = \sum_{l=0}^{k} \frac{(-1)^l}{l!} d^l_T \circ S^*l \) is the variational derivative operator

\[
\delta : \bigwedge^1(T^kM) \to \bigwedge^1(\tau_{2k}).
\]

The one-form \( \alpha_L \) vanishes on every point of the image of \( \gamma_{0,2k} \) for each curve \( \gamma_0 \) solution of Hamilton’s Variational Principle for \( L \); the set of points \( D_L = \tilde{\alpha}_L^{-1}(0) = \{ u \in T^{2k}M, \tilde{\alpha}_L(u) = 0 \} \) is the submanifold of \( T^{2k}M \) representing the 2\( k \)-th-order Euler-Lagrange differential equations for \( L \).

The Poincaré-Cartan form \( \theta_L \) is obtained from \( \delta \) and \( dL \) by decomposing \( \delta \) as \( \delta = \text{id} - d_T \circ S \) [3, 9] with

\[
S = \sum_{l=1}^{k} \frac{(-1)^{l-1}}{l!} d^{l-1}_T \circ S^*l
\]

and

\[
\theta_L = S(dL) \in \bigwedge^1(T^{2k-1}M).
\]

This one-form \( \theta_L \) happens to be \( k \)-semibasic, \( S^*k(\theta_L) = 0 \), so that it is in fact a one-form along the projection \( \tau_{2k-1}^{k-1} \), \( \tilde{\theta}_L : T^{2k-1}M \to T^*(T^{k-1}M) \), showing that the phase space for this dynamical system is the cotangent bundle \( T^*(T^{k-1}M) \) [9] and the Legendre-Ostrogradsky transformation is but \( \theta_L \) itself.
5. Variational derivative versus exterior differential and Helmholtz conditions for the Inverse Problem

In the two previous sections we have developed a construction of the variational derivative as a bundle operator associated to the exterior differential on the manifold $\mathcal{C}$ of curves over $M$. Beginning with an “integral” function $A : \mathcal{C} \to \mathbb{R}$, given by $A(\gamma) = \int_I \gamma^*(L) dt$, for $L \in C^\infty(T^k M)$, its exterior differential on a subset $\mathcal{C}_I$ of fixed endpoints takes the form

$$i(X)dA = \int_I [i(X_k)dL]dt = \int_I [i(X_{2k}\delta(dL))dt,$$

where $X \in T_\gamma \mathcal{C}_I$.

Although both integral expressions for $i(X)dA$ are equivalent, only the last one has the property that the integrand $\delta(dL)|_{\gamma_0}$ vanishes when $dA|_{\gamma_0} = 0$. For any other alternative integral expression for $dA$, the integrand is a total time derivative whose integral vanishes, but not the integrand, when $dA = 0$. Therefore this property makes possible to pick out the variational derivative as a distinguished integral expression for $dA$ among others.

Let us consider now an “integral” one-form $B$ on $\mathcal{C}_I$, with associated one-form $\beta$ on $T^k M$, $\beta \in \bigwedge^1(T^k M)$ as follows:

$$i(X)B(\gamma) \equiv \int_I [i(X_k)\beta]dt \quad \forall X \in T_\gamma \mathcal{C}_I \equiv X(\gamma)$$

For example, the exterior differential of an “integral” function $A$ on $\mathcal{C}_I$ is an “integral” one-form $dA$. Let us compute the exterior differential of $B$; the use of the expression

$$dB(X,Y)(\gamma) = i(X(\gamma))d[i(Y)\beta] - i(Y(\gamma))d[i(X)\beta] - i([X,Y])(\gamma))B$$

with $X, Y \in X(\mathcal{C}_I)$ has the problem of dealing with vector fields on the infinite dimensional manifold $\mathcal{C}_I$, and their Lie brackets. As $dB(X,Y)(\gamma)$ depends only on the values of $X$ and $Y$ on $\gamma$, we can avoid this problem by considering a special case for $X$ and $Y$, namely when they are associated to vector fields $X,Y$ on the finite dimensional manifold $M$, by

$$X(\gamma) \equiv X_\gamma \in X(\gamma) \text{ with } X_\gamma(t) = X(\gamma(t))$$

and similarly for $Y$ with $Y$. The value of $X$ on a point $\gamma$ of $\mathcal{C}_I$ is but the restriction of the vector field $X \in X(M)$ to the image of $\gamma$. Of course, in order to do that we must choose these vector fields with the appropriate boundary conditions, and develop the analysis piecewise when the curve $\gamma$ has self-intersections.

In this case we have that $i(Y)B = A_Y : \mathcal{C}_I \to \mathbb{R}$ is given by

$$A_Y(\gamma) = \int_I \gamma^*_k[i(Y_k)\beta]dt$$
and similarly for $A_X$. The Lie bracket can now be computed. For every $L \in C^\infty(T^k M)$, $A_L : \mathcal{C}_I \to \mathbb{R}$ is

$$A_L(\gamma) = \int_I \gamma^*_k(L) dt.$$ 

Then,

$$[\mathcal{X}, \mathcal{Y}](A_L)(\gamma) = \int_I (i(X_\gamma)[dA_L]_Y - i(Y_\gamma)[dA_L]_X) dt = \int_I i(X_\gamma)[dA_L]_Y dt - \int_I i(Y_\gamma)[dA_L]_X dt = \int_I \gamma^*_k\{i([X, Y]_k)dL\} dt.$$

so that $i([\mathcal{X}, \mathcal{Y}]B)(\gamma) = \int_I \gamma^*_k\{i([X, Y]_k)\beta\} dt$. Finally, $dB(X_\gamma, Y_\gamma)$ takes the form

$$dB(X_\gamma, Y_\gamma) = \int_I \gamma^*_k\{i(X_k)d[i(Y_k)\beta] - i(Y_k)d[i(X_k)\beta] - i([X, Y]_k)\beta\} dt = \int_I d\beta(X_{\gamma_k}(t), Y_{\gamma_k}(t)) dt$$

At this point our aim is to obtain a bundle operator in order to reexpress the “integral” two-form $dB$ in such a way that the vanishing of $dB$ be equivalent to the vanishing of the integrand, as we made for integral functionals. To do that, let us consider two vectors $X, Y \in T_\gamma \mathcal{C}_I$ and an arbitrary function $f \in C^\infty(M)$. If $\tilde{f} = \gamma^*(f)$, then

$$dB(\tilde{f}X, Y) = \int_I d\beta((\tilde{f}X)_k(t), Y_k(t)) dt = \int_I i(Y_k(t))\{\sum_{l=0}^k \frac{1}{l!} \frac{d^l \tilde{f}}{dt^l} i(S_l^k(X_k))d\beta\} dt = \int_I \tilde{f} \sum_{l=0}^k (-1)^l \frac{d^l \tilde{f}}{dt^l} i(Y_k(t)) i(X_k(t)) S^*_l(\beta) dt + \text{boundary terms}$$

where $S^*_1(\omega)(X, Y) \equiv \omega(SX, Y)$. Note that $S^*_1(\omega)$ is a two covariant tensor but not a skewsymmetric one. Boundary terms vanish because of the fixed endpoints conditions on $\mathcal{C}_I$, and commuting the time derivative $\frac{d}{dt}$ with the inner contractions $i(Y_k)$ and $i(X_k)$ leads to

$$dB(\tilde{f}X, Y) = \int_I \tilde{f} \sum_{l=0}^k \frac{(-1)^l}{l!} i(Y_{k+l}(t)) i(X_{k+l}(t)) [d_{T^k} \circ S^*_l(\beta)] dt$$

so that $dB(\gamma) = 0$ if and only if $\delta_1(d\beta)|_{\gamma_{2k}} = 0$, where (Ibitcoin and López, 1990)

$$\delta_1 \equiv \sum_{l=0}^k \frac{(-1)^l}{l!} d_{T^k} \circ S^*_l : \bigwedge^2 (T^k M) \to \bigwedge^2 (T^{2k} M).$$
An equivalent operator would be obtained from \( dB(X, \bar{f}Y) \),

\[
\delta_2 \equiv \sum_{l=0}^{k} \frac{(-1)^l}{l!} d^l_T \circ S^*_l
\]

with \( S^*_l(\omega)(X, Y) \equiv \omega(X, SY) \). However, \( dB(X, Y) \) is skewsymmetric, so that only the skewsymmetric part of \( \delta_1 \) or \( \delta_2 \) is nontrivial, the symmetric part giving rise to a total time derivative term, whose integral vanishes. The variational derivative operator can be defined acting on two-forms on \( T^k M \) as a map \( \delta^A : \bigwedge^2 (T^k M) \to \bigwedge^2 (T^{2k} M) \), by means of

\[
\delta^A \equiv \frac{1}{2} (\delta_1 + \delta_2) = \text{id} + \frac{1}{2} \sum_{l=1}^{k} \frac{(-1)^l}{l!} d^l_T \circ \{ S^*_1 + S^*_2 \}
\]

Note that tensor \( (S^*\omega)(X, Y) = \omega(SX, SY) + \omega(X, SY) \) is skewsymmetric, as well as \( S^*_l \omega \), but it is different from \( S^*_1 + S^*_2 \); for example

\[
S^*_2(\omega)(X, Y) = \omega(S^2 X, Y) + 2\omega(SX, SY) + \omega(X, S^2 Y)
\]

\[= \{ S^*_1 + S^*_2 \} \omega(X, Y) + 2S^*_1 S^*_2 \omega(X, Y) \]

In the case of \( \beta \) being a semibasic one-form, \( S^*\beta = 0 \), all the intermediate terms on \( S^*d\beta \) are zero because of \( d\beta \) vanishing on two vertical vectors, and therefore, for this particular case we can rewrite

\[
\delta^A(d\beta) = d\beta + \frac{1}{2} \sum_{l=1}^{k} \frac{(-1)^l}{l!} d^l_T \circ S^*(d\beta).
\]

The case of semibasic one-forms is very relevant for the Inverse Problem of Lagrangian Mechanics because, as we have seen, Euler-Lagrange equations are geometrically represented by a semibasic one-form.

Equivalence between the vanishing of \( dB(\gamma) \) and that of \( \delta^A(d\beta) \mid_{\gamma_{2k}} \) means that an “integral” type one-form \( B \) on \( \mathcal{C}_I \) is closed if and only if its associated one-form \( \beta \) on \( T^k M \) is \( \delta \)-closed, i.e., \( \delta^A(d\beta) = 0 \). In particular, property \( d^2 = 0 \) on \( \mathcal{C}_I \) is equivalent to \( (\delta^A \circ d) \circ (\delta \circ d) \mid_{C^\infty(T^* M)} = 0 \) for arbitrary \( k \).

It is now clear that the former calculus can be applied to higher degree forms of “integral” type on \( \mathcal{C}_I \), and taking always the fully skew-symmetric part of the resulting variational derivative operator, we would get a map \( \delta^A : \bigwedge^m (T^k M) \to \bigwedge^m (T^{2k} M) \) defined by

\[
\delta^A \equiv \text{id} + \frac{1}{m} \sum_{l=1}^{k} \frac{(-1)^l}{l!} d^l_T \circ \{ S^*_1 + S^*_2 + \cdots + S^*_m \}.
\]

As a consequence of the former analysis, the Euler-Lagrange equation form \( \alpha_L = \delta(dL) \) for a given Lagrangian function \( L \in C^\infty(T^k M) \) is \( \delta \)-closed, i.e., such that \( \delta^A(d\alpha_L) = \).
0. The converse is also locally true; if a system $D$ of $r$-th order differential equations $D \subset T^r M$ is presented through a semibasic one-form $\alpha : T^r M \to T^* M$, $D = \alpha^{-1}(0)$, which is $\delta$-closed, then there exists at least locally a Lagrangian function for $\alpha$, $\alpha = \delta(dL)$. An explicit geometric construction of the Lagrangian from $\alpha$ can be found in a previous paper by Ibort and López, (1990).

The Inverse Problem of Lagrangian Mechanics goes one step further, and asks when a system of differential equations can be presented in at least one of its equivalent representations in such a way that it corresponds to the Euler-Lagrangian equations for some Lagrangian. Therefore, it deals with all the possible representations of a given system of differential equations $D$, $(\alpha_1^{-1}(0) = \alpha_2^{-1}(0) = \ldots = D \subset T^r M)$, and the problem is to check all these representations in order to find one which is $\delta$-closed.

It is usual in the literature to consider just a restricted inverse problem, dealing with systems of second order differential equations presented in normal form, (Santilli, 1978), i.e., through a section $\sigma$ of $\tau_2^1$. As we are treating the higher order case too, we will consider systems of $2k$-th order differential equations $D \subset T^{2k} M$ presented in normal form $D = \text{Im} \sigma \, \sigma : T^{2k-1} M \to T^{2k} M$. There is a natural representation for these systems which will be called the vectorial representation, determined as follows: let $\Gamma_\sigma = i_{1,2k-1} \circ \sigma \in \mathfrak{X}(T^{2k-1} M)$ be the $2k$-th ODE vector field, $S(\Gamma_\sigma) = \Delta$, associated to the system $D$; in local coordinates, if $D$ is given by

$$\frac{d^{2k} x^i}{dt^{2k}} = \phi^i(x, \frac{dx}{dt}, \ldots, \frac{d^{2k-1} x}{dt^{2k-1}})$$

the associated vector field is

$$\Gamma_\sigma = \sum_{l=1}^{2k-1} x^i_l \frac{\partial}{\partial x^i_{l-1}} + \phi^i(x, x_1, \ldots, x_{2k-1}) \frac{\partial}{\partial x^i_{2k-1}}.$$ 

There is a first order derivative operator $\mathbb{D}_\sigma : C^\infty(M) \to C^\infty(T^{2k} M)$, given by $\mathbb{D}_\sigma(f) = [d_T - \tau_{2k}^{-1} \circ \Gamma_\sigma](d_T^{-1} f)$, with associated vector field along $\tau_{2k}, Z_\sigma$. In local coordinates, for a coordinate function $x^i$,

$$\mathbb{D}_\sigma(x^i) = x^i_{2k} - \phi^i$$

so that $Z_\sigma = [x^i_{2k} - \phi^i] \partial_{x^i} \in \mathfrak{X}(\tau_{2k})$. This vector field is the vectorial representation for $D$, the inverse image by the map $Z_\sigma$ of the zero section of $\tau$ is the submanifold $D$ of $T^{2k} M$ representing the system of differential equations.

Many other representations can be obtained from the vectorial representation $Z_\sigma$. For example, let us consider a two covariant tensor $g$ along $\tau_k$, $g : T^k M \to T^* M \times T^* M$; for every $u \in T^k M$, $g(u)$ can be seen as a map $g(u) : T_x M \to T_x^* M$, with $x = \tau_k(u)$, defined by contraction with the first argument

$$[g(u)(v)](w) = g(u)(v, w) \quad \forall v, w \in T_x M$$
In local coordinates, \( g(u) = g_{ij}(u)dx^i \otimes dx^j \) and \( g(u)(v) = g_{ij}(u)v^i dx^j \). With this tensor \( g \), assumed to be nonsingular, and the vectorial representation \( Z_\sigma \) of \( D \) we obtain a one-form representation \( \alpha_g : T^{2k} M \to T^* M \) by \( \alpha_g(u) = g(\tau_{2k}^k(u))(Z_\sigma(u)) \). In this way we can obtain a family of possible one-form representations of \( D \) which are affine in the higher order derivatives \( x_{2k}^i \)

\[
\alpha_g(x, x_1, \ldots, x_{2k}) = g_{ij}(x, x_1, \ldots, x_k)[x_{2k}^i - \phi^i]dx^j
\]

The possible representations of \( D \) which are of Euler Lagrange form are necessarily among those \( \alpha_g \), because the Euler Lagrange equations for \( L \in C^\infty(T^k M) \) are affine in the higher order derivatives and the coefficients are functions on \( T^k M \),

\[
\frac{\partial^2 L}{\partial x_k^i \partial x_k^j} \frac{d^{2k}x^i}{dt^{2k}} + \cdots = 0.
\]

For a given tensor \( g \), the associated one-form \( \alpha_g \) representing the system of differential equations is on Euler-Lagrange form iff

\[
\delta^A(d\alpha_g) = 0
\]

where \( \alpha_g \) is seen as a one-form on \( T^{2k} M \) by pull-back.

The latter geometric condition, considered as an equation on the unknown \( g \), for a given system of differential equations \( Z_\sigma \), is a geometric version of the system of partial differential equations known as Helmholtz conditions for the Inverse Problem of Lagrangian Mechanics, and is valid for general higher order differential equations.
6. Coordinate expressions and example

Helmholtz conditions on the Inverse Problem, for a given system of second order differential equations \( x_i^2 = \phi^i(x, x_1) \), are a system of partial differential equations on an unknown matrix \( g_{ij}(x, x_1) \), such that if there is a solution \( g \), the equivalent system of differential equations \( g_{ij}x_i^2 = g_{ij}\phi^i \) are the Euler-Lagrange equations for a Lagrangian, which can be determined from \( g \) and \( \phi^i \). The conditions on \( g \) take the form (See Santilli, 1978):

(a) \( g_{ij} = g_{ji} \)
(b) \( \frac{\partial g_{ij}}{\partial v^k} = \frac{\partial g_{ik}}{\partial v^j} \)
(c) \( \Gamma(g_{ij}) + \frac{1}{2} g_{kj}\frac{\partial \phi^k}{\partial v^i} + \frac{1}{2} g_{ik}\frac{\partial \phi^k}{\partial v^j} = 0 \)

with \( \Gamma = v^i\partial/\partial x^i + \phi^j\partial/\partial v^i \) the SODE vector field associated to the system, and

(d) \( g_{ik}\left[ \frac{\partial \phi^k}{\partial x^j} + \frac{1}{4} \frac{\partial \phi^k}{\partial v^i} \frac{\partial \phi^j}{\partial v^i} - \frac{1}{2} \left( \frac{\partial \phi^k}{\partial v^j} \right) \right] + \{i \rightarrow j\} = 0 \),

where \( \{i \rightarrow j\} \) means the interchange of the index.

According to the analysis developed in the preceding section, these equations must be but the different components of the two form \( \delta A(d\alpha) = 0 \), for the one form \( \alpha = g_{ij}[x_i^2 - \phi^j(x, x_1)]dx^i \). Let us develop the coordinate expression for \( \delta A(d\alpha) \) for a comparison with the former conditions. First of all, in order to simplify the calculus, note that if \( \delta A(d\alpha) = 0 \), then \( d\alpha = \frac{1}{2} d_T S^*(d\alpha) - \frac{1}{4} d_T S^{*2}(d\alpha) \), from which it can be proved that \( S^{*2}(d\alpha) = 0 \), taking into account the property \([S^*, d_T]|_{\Lambda^2(T^kM)} = 2 id_{\Lambda^2(T^kM)} \) [5]. Therefore we have two conditions for \( d\alpha \):

\[ S^{*2}(d\alpha) = 0 \quad \text{and} \quad d\alpha - \frac{1}{2} d_T S^*(d\alpha) = 0 \]

The coordinate expression of \( d\alpha \) is

\[ d\alpha = g_{ij}dx_2^i \wedge dx^j + \left[ x^i_2 \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial X_j}{\partial x^k} \right] dx_1^i \wedge dx^j + \left[ x^i_2 \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial X_j}{\partial x^k} \right] dx_1^k \wedge dx^j \]

where \( \chi_i \equiv g_{ij}\phi^j \). The first condition, \( S^{*2}(d\alpha) = 0 \), is very simple

\[ g_{ij}dx^i \wedge dx^j = 0, \]

and gives condition (a) of symmetry on \( g \). The second condition, \( d\alpha - \frac{1}{2} d_T S^*(d\alpha) = 0 \), takes the form

\[ d\alpha - \frac{1}{2} d_T S^* d\alpha = \left[ \frac{1}{2} x^i_2 \frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2} \frac{\partial \chi_j}{\partial x^k} - d_T g_{jk} + \frac{1}{2} x^i_2 \frac{\partial g_{ik}}{\partial x^j} - \frac{1}{2} \frac{\partial \chi_k}{\partial x^j} \right] dx_1^k \wedge dx^j + \]
\[
\left[ x_i^2 \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial x_j}{\partial x^k} - \frac{1}{2} x^3 \frac{\partial g_{ij}}{\partial x^1} + \frac{1}{2} d_T \left( \frac{\partial x_j}{\partial x^1} \right) \right] dx^k \wedge dx^i = 0 ,
\]

from which we obtain
\[
\frac{\partial g_{ij}}{\partial x^i} = \frac{\partial x^j}{\partial x^i}
\]
and
\[
\frac{\partial g_{ij}}{\partial x^j} + \frac{1}{2 x^1} \frac{\partial^2 x_j}{\partial x^i \partial x^1} - \{ j \to k \} = 0
\]

The fact that these conditions are necessary for Euler-Lagrange equations is easily checked, identifying \( g_{ij} = \frac{\partial^2 L}{\partial x^i \partial x^j} \) and \( \chi_i = \frac{\partial L}{\partial x^i} - x^j \frac{\partial^2 L}{\partial x^i \partial x^j} \) from the coordinate expression of Euler-Lagrange equations for a Lagrangian \( L \). In order to compare with Helmholtz conditions we must substitute \( \chi_j \) by \( g_{ij} \phi^j \), so obtaining
\[
\Gamma(g_{ij}) + \frac{1}{2} g_{jk} \frac{\partial \phi^k}{\partial x^i} + \frac{1}{2} g_{ik} \frac{\partial \phi^k}{\partial x^1} = 0
\]
and
\[
M_{ijk} = \frac{\partial g_{ij}}{\partial x^k} + \frac{1}{2} g_{jk} \frac{\partial \phi^l}{\partial x^i} + \frac{1}{2} g_{jl} \frac{\partial \phi^l}{\partial x^i \partial x^1} - \{ j \to k \} = 0
\]

We will recover from the two last conditions the last one of Helmholtz conditions, (d), by \( \phi^i M_{ijk} - P_{jk} = 0 \).

In a similar way we could obtain the coordinate expression of Helmholtz conditions on \( g \) for higher order differential equations, but in order to simplify the expressions we are going to consider a particular example of a fourth order differential equation in one variable, and with no dependence on third order derivative
\[
\frac{d^4 x}{dt^4} = \phi \left( x, \frac{dx}{dt}, \frac{d^2 x}{dt^2} \right)
\]

The one form representation for this equation is
\[
\alpha = g[x_A - \phi] dx
\]
with $g$ an arbitrary function on $T^2\mathbb{R}$. From

$$d\alpha = g \left[ dx_4 - \frac{\partial \phi}{\partial x_1} dx_1 - \frac{\partial \phi}{\partial x_2} dx_2 \right] \wedge dx + (x_4 - \phi) \left[ \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 \right] \wedge dx$$

we obtain

$$\delta^A (d\alpha) = d\alpha - \frac{1}{2} d_T S^* (d\alpha) + \frac{1}{4} d_T^2 S^* (d\alpha) - \frac{1}{12} d_T^3 S^* (d\alpha)$$

$$= -2 d_T g dx_3 \wedge dx + g \left[ d_T \left( \frac{\partial \phi}{\partial x_2} \right) - \frac{\partial \phi}{\partial x_1} \right] dx_1 \wedge dx = 0$$

Therefore $g$ must be constant, $d_T g = 0$, and conditions on $\phi$ appear, $\frac{\partial^2 \phi}{\partial x_2^2} = 0$ and

$$x_2 \frac{\partial^2 \phi}{\partial x_2 \partial x_1} + x_1 \frac{\partial^2 \phi}{\partial x_2 \partial x} - \frac{\partial \phi}{\partial x_1} = 0,$$

so that $\phi$ must be linear on $x_2$,

$$\phi = A(x, x_1) x_2 + B(x, x_1)$$

while the functions $A$ and $B$ must satisfy

$$x_1 \frac{\partial A}{\partial x} - \frac{\partial B}{\partial x_1} = 0.$$

If these conditions on $\phi$ are satisfied, there exists a Lagrangian function for the differential equation and this Lagrangian is essentially unique, up to trivial equivalence by adding a total time derivative. Note that, contrarily to the case of second order differential equations, the one-dimensional problem is not always Lagrangian for higher order differential equations.

Let us consider two functions $A(x, x_1)$ and $U(x)$, and the differential equation

$$\frac{d^4 x}{dt^4} = A(x, x_1) x_2 + B(x, x_1)$$

with $B = \int [x_1 \frac{\partial A}{\partial x}] dx_1 + U(x)$, so that condition $x_1 \frac{\partial A}{\partial x} = \frac{\partial B}{\partial x_1}$ are fulfilled. The Lagrangian function for this differential equation can be obtained by using the method developed in the paper by Ibort and López [8]. Taking $g = 1$ (any other constant would give a proportional Lagrangian), we have

$$\alpha = |x_4 - Ax_2 - B| dx .$$

The Cartan two form $\omega_L$ is obtained from $\alpha$ by

$$\alpha = dL - d_T \theta_L \quad d\alpha = d_T \omega_L$$
\[ d\alpha = d_T \left[ \text{id} - \frac{1}{2} d_T S^* + \frac{1}{6} d_T^2 S^{*2} \right] \left[ \frac{1}{2} S^* \right] (d\alpha) \]

so that

\[ \omega_L = \left[ \text{id} - \frac{1}{2} d_T S^* + \frac{1}{6} d_T^2 S^{*2} \right] \left[ \frac{1}{2} S^* \right] (d\alpha) \]

In coordinates, it takes the form

\[ \omega_L = dx_3 \wedge dx + dx_2 \wedge dx_1 - Adx_1 \wedge dx \]

which is obviously closed. The Poincaré-Cartan one-form \( \theta_L \) is not uniquely determined, the ambiguity been related to the trivial ambiguity on the Lagrangian. Som \( L = L_0 + d_T \rho \) implies \( \theta_L = \theta_0 + d\rho \). In this case, the Poincaré-Cartan form is given by

\[ \theta_L = \left[ -x_3 + \int A dx_1 \right] dx + x_2 dx_1 + d\rho \]

The Lagrangian is now obtained from

\[ dL = \alpha + d_T \theta_L = x_2 dx_2 + \left[ \int A dx_1 \right] dx_1 + \left[ - \int x_1 \frac{\partial A}{\partial x} dx_1 - U(x) + x_1 \int \frac{\partial A}{\partial x} dx_1 \right] dx + d_T d\rho \]

so that finally

\[ L = \frac{1}{2} x_2^2 + \int \left[ \int A dx_1 \right] dx_1 - \int U dx + d_T \rho \]

is given explicitly from the data \( A(x, x_1) \) and \( U(x) \).

**References**